Mediterranean Mathematics Olympiad 2003

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by Amir Hossein, erdugan

1   Prove that the equation $x^2 + y^2 + z^2 = x + y + z + 1$ has no rational solutions.

2   In a triangle ABC with $BC = CA + \frac{1}{2}AB$, point P is given on side AB such that $BP : PA = 1 : 3$. Prove that $\angle CAP = 2\angle CPA$.

3   Let a, b, c be non-negative numbers with $a + b + c = 3$. Prove the inequality

$$\frac{a}{b^2 + 1} + \frac{b}{c^2 + 1} + \frac{c}{a^2 + 1} \geq \frac{3}{2}.$$

4   Consider a system of infinitely many spheres made of metal, with centers at points $(a, b, c) \in \mathbb{Z}^3$. We say that the system is stable if the temperature of each sphere equals the average temperature of the six closest spheres. Assuming that all spheres in a stable system have temperatures between $0°C$ and $1°C$, prove that all the spheres have the same temperature.