Mediterranean Mathematics Olympiad 2008

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by bluecarneal

1 Let $ABCDEF$ be a convex hexagon such that all of its vertices are on a circle. Prove that AD, BE and CF are concurrent if and only if $\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1$.

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2 Determine whether there exist two infinite point sequences $A_1, A_2, \dots$ and $B_1, B_2, \dots$ in the plane, such that for all i, j, k with $1 \leq i < j < k$,
(i) B_k is on the line that passes through A_i and A_j if and only if $k = i + j$.
(ii) A_k is on the line that passes through B_i and B_j if and only if $k = i + j$.
(Proposed by Gerhard Woeginger, Austria)

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3 Let n be a positive integer. Calculate the sum $\sum_{k=1}^{n} \sum_{1 \leq i_1 < \dots < i_k \leq n} \frac{2^k}{(i_1+1)(i_2+1)\dots(i_k+1)}$

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4 The sequence of polynomials (a_n) is defined by $a_0 = 0$, $a_1 = x + 2$ and $a_n = a_{n-1} + 3a_{n-1}a_{n-2} + a_{n-2}$ for $n > 1$.
(a) Show for all positive integers k, m: if k divides m then a_k divides a_m.
(b) Find all positive integers n such that the sum of the roots of polynomial a_n is an integer.
