Mediterranean Mathematics Olympiad 2009

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by bluecarneal

1 Determine all integers $n \geq 1$ for which there exists n real numbers $x_1, \ldots, x_n$ in the closed interval $[-4, 2]$ such that the following three conditions are fulfilled:
- the sum of these real numbers is at least n.
- the sum of their squares is at most $4n$.
- the sum of their fourth powers is at least $34n$.
(Proposed by Gerhard Woeginger, Austria)

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2 Let ABC be a triangle with $90^\circ \neq \angle A \neq 135^\circ$. Let D and E be external points to the triangle ABC such that DAB and EAC are isoscele triangles with right angles at D and E. Let $F = BE \cap CD$, and let M and N be the midpoints of BC and DE, respectively.

Prove that, if three of the points A, F, M, N are collinear, then all four are collinear.

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3 Decide whether the integers $1, 2, \ldots, 100$ can be arranged in the cells $C(i, j)$ of a 10×10 matrix (where $1 \leq i, j \leq 10$), such that the following conditions are fullfiled:
i) In every row, the entries add up to the same sum S.
ii) In every column, the entries also add up to this sum S.
iii) For every $k = 1, 2, \ldots, 10$ the ten entries $C(i, j)$ with $i - j \equiv k \bmod 10$ add up to S.
(Proposed by Gerhard Woeginger, Austria)

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4 Let x, y, z be positive real numbers. Prove that

$$\sum_{cyclic} \frac{xy}{xy + x^2 + y^2} \leq \sum_{cyclic} \frac{x}{2x + z}$$

(Proposed by efket Arslanagi, Bosnia and Herzegovina)
