Mediterranean Mathematics Olympiad 2012

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by djb86, sqing, Math-lover123

1 For a real number $\alpha > 0$, consider the infinite real sequence defined by $x_1 = 1$ and

$$\alpha x_n = x_1 + x_2 + \cdots + x_{n+1} \qquad \text{for } n \geq 1.$$

Determine the smallest α for which all terms of this sequence are positive reals.
(Proposed by Gerhard Woeginger, Austria)

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2 In an acute $\triangle ABC$, prove that

$$\frac{1}{3}\left(\frac{\tan^2 A}{\tan B \tan C} + \frac{\tan^2 B}{\tan C \tan A} + \frac{\tan^2 C}{\tan A \tan B}\right)$$
$$+3\left(\frac{1}{\tan A + \tan B + \tan C}\right)^{\frac{2}{3}} \geq 2.$$

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3 Consider a binary matrix M(all entries are 0 or 1) on r rows and c columns, where every row and every column contain at least one entry equal to 1. Prove that there exists an entry $M(i,j) = 1$, such that the corresponding row-sum $R(i)$ and column-sum $C(j)$ satisfy $rR(i) \geq cC(j)$.
(Proposed by Gerhard Woeginger, Austria)

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4 Let O be the circumcenter,R be the circumradius, and k be the circumcircle of a triangle ABC
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Let k_1 be a circle tangent to the rays AB and AC, and also internally tangent to k.
Let k_2 be a circle tangent to the rays AB and AC , and also externally tangent to k. Let A_1 and A_2 denote the respective centers of k_1 and k_2.
Prove that: $(OA_1 + OA_2)^2 - A_1A_2^2 = 4R^2$.
