Danube Mathematical Olympiad 2005

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1 Prove that the equation $4x^3-3x+1=2y^2$ has at least 31 solutions in positive integers x and y with $x\leq 2005$.

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2 Prove that the sum:

$$S_n=\binom{n}{1}+\binom{n}{3}\cdot 2005+\binom{n}{5}\cdot 2005^2+\ldots=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2k+1}\cdot 2005^k$$

is divisible by 2^{n-1} for any positive integer n.

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3 Let $\mathcal{C}$ be a circle with center O, and let A be a point outside the circle. Let the two tangents from the point A to the circle $\mathcal{C}$ meet this circle at the points S and T, respectively. Given a point M on the circle $\mathcal{C}$ which is different from the points S and T, let the line MA meet the perpendicular from the point S to the line MO at P.

Prove that the reflection of the point S in the point P lies on the line MT.

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4 Let k and n be positive integers. Consider an array of $2\left(2^n-1\right)$ rows by k columns. A 2-coloring of the elements of the array is said to be *acceptable* if any two columns agree on less than 2^n-1 entries on the same row.

Given n, determine the maximum value of k for an acceptable 2-coloring to exist.
