Danube Mathematical Olympiad 2010

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by littletush, Maxim Bogdan, paul1703

1 Determine all integer numbers $n \geq 3$ such that the regular n-gon can be decomposed into isosceles triangles by non-intersecting diagonals.

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2 Given a triangle ABC, let A', B', C' be the perpendicular feet dropped from the centroid G of the triangle ABC onto the sides BC, CA, AB respectively. Reflect A', B', C' through G to A'', B'', C'' respectively. Prove that the lines AA'', BB'', CC'' are concurrent.

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3 All sides and diagonals of a convex n-gon, $n \geq 3$, are coloured one of two colours. Show that there exist $\left[\frac{n+1}{3}\right]$ pairwise disjoint monochromatic segments.

(Two segments are disjoint if they do not share an endpoint or an interior point).

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4 Let p be a prime number of the form $4k+3$. Prove that there are no integers w, x, y, z whose product is not divisible by p, such that:

$$w^{2p} + x^{2p} + y^{2p} = z^{2p}.$$

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5 Let $n \geq 3$ be a positive integer. Find the real numbers $x_1 \geq 0, \ldots, x_n \geq 0$, with $x_1 + x_2 + \ldots + x_n = n$, for which the expression

$$(n-1)(x_1^2 + x_2^2 + \ldots + x_n^2) + nx_1x_2 \ldots x_n$$

takes a minimal value.
