Moldova Team Selection Test 2003

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by DreamTeam

Day 1

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1 Each side of an arbitrarly triangle is divided into 2002 congruent segments. After that, each vertex is joined with all "division" points on the opposite side.
Prove that the number of the regions formed, in which the triangle is divided, is divisible by 6.

Proposer: **Dorian Croitoru**

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2 The positive reals x, y and z are satisfying the relation $x+y+z \geq 1$. Prove that: $\frac{x\sqrt{x}}{y+z}+\frac{y\sqrt{y}}{z+x}+\frac{z\sqrt{z}}{x+y} \geq \frac{\sqrt{3}}{2}$

Proposer: **Baltag Valeriu**

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3 Let $ABCD$ be a quadrilateral inscribed in a circle of center O. Let M and N be the midpoints of diagonals AC and BD, respectively and let P be the intersection point of the diagonals AC and BD of the given quadrilateral .It is known that the points O, M, Np are distinct. Prove that the points O, N, A, C are concyclic if and only if the points O, M, B, D are concyclic.

Proposer: **Dorian Croitoru**

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4 Prove that the equation $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{abc}=\frac{12}{a+b+c}$ has infinitely many solutions (a, b, c) in natural numbers.

Day 2

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1 Let $n>0$ be a natural number. Determine all the polynomials of degree $2n$ with real coefficients in the form $P(X)=X^{2n}+(2n-10)X^{2n-1}+a_2X^{2n-2}+...+a_{2n-2}X^2+(2n-10)X+1$, if it is known that all the roots of them are positive reals.

Proposer: **Baltag Valeriu**

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2 Consider the triangle ABC with side-lenghts equal to a, b, c. Let $p=\frac{a+b+c}{2}$, R-the radius of circumcircle of the triangle ABC, r-the radius of the incircle of the triangle ABC and let l_a, l_b, l_c be the lenghts of bisectors drawn from A, B and C, respectively, in the triangle ABC. Prove that: $l_a l_b+l_b l_c+l_c l_a \leq p\sqrt{3r^2+12Rr}$

Proposer: **Baltag Valeriu**

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3 The sides $[AB]$ and $[AC]$ of the triangle ABC are tangent to the incircle with center I of the $\triangle ABC$ at the points M and N, respectively. The internal bisectors of the $\triangle ABC$ drawn form B and C intersect the line MN at the points P and Q, respectively. Suppose that F is the intersection point of the lines CP and BQ. Prove that $FI \perp BC$.

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4 On the fields of a chesstable of dimensions $n \times n$, where $n \geq 4$ is a natural number, are being put coins. We shall consider a *diagonal* of table each diagonal formed by at least 2 fields. What is the minimum number of coins put on the table, s.t. on each column, row and diagonal there is at least one coin? Explain your answer.

Day 3

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1 Let $n \in N^*$. A permutation $(a_1, a_2, ..., a_n)$ of the numbers $(1, 2, ..., n)$ is called *quadratic* iff at least one of the numbers $a_1, a_1+a_2, ..., a_1+a_2+a+...+a_n$ is a perfect square. Find the greatest natural number $n \leq 2003$, such that every permutation of $(1, 2, ..., n)$ is quadratic.

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2 Let $a_1, a_2, ..., a_{2003} \geq 0$, such that $a_1 + a_2 + ... + a_{2003} = 2$ and $a_1a_2 + a_2a_3 + ... + a_{2003}a_1 = 1$. Determine the minimum and maximum value of $a_1^2 + a_2^2 + ... + a_{2003}^2$.

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3 Consider a point M found in the same plane with the triangle ABC, but not found on any of the lines AB, BC and CA. Denote by S_1, S_2 and S_3 the areas of the triangles AMB, BMC and CMA, respectively. Find the locus of M satisfying the relation: $(MA^2 + MB^2 + MC^2)^2 = 16(S_1^2 + S_2^2 + S_3^2)$

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4 A square-table of dimensions $n \times n$, where $n \in N^*$, is filled arbitrarily with the numbers $1, 2, ..., n^2$ such that every number appears on the table exactly one time. From each row of the table is chosen the least number and then denote by x the biggest number from the numbers chosen. From each column of the table is chosen the least number and then denote by y the biggest number from the numbers chosen. The table is called *balanced* iff $x = y$. How many balanced tables we can obtain?