Moldova Team Selection Test 2005

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by darij grinberg, Sasha, Tales

Day 1

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1 Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let AB be a diameter of the circle Γ; assume that this diameter AB is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same halfplane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A.

Prove that the reflection of the point G in the line AB lies on the line CF.

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2 Let a, b, c be positive reals such that $a^4+b^4+c^4=3$. Prove that $\sum\frac{1}{4-ab}\leq 1$, where the $\sum$ sign stands for cyclic summation.

Alternative formulation: For any positive reals a, b, c satisfying $a^4+b^4+c^4=3$, prove the inequality
$\frac{1}{4-bc}+\frac{1}{4-ca}+\frac{1}{4-ab}\leq 1$.

3

$$A=3\sum_{m=1}^{n^2}(\frac{1}{2}-\{\sqrt{m}\})$$

where n is an positive integer. Find the largest k such that n^k divides $[A]$.

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4 n is a positive integer, K the set of polynoms of real variables $x_1,x_2,...,x_{n+1}$ and $y_1,y_2,...,y_{n+1}$, function $f:K\to K$ satisfies

$$f(p+q)=f(p)+f(q),\quad f(pq)=f(p)q+pf(q),\quad (\forall)p,q\in K.$$

If $f(x_i)=(n-1)x_i+y_i,\quad f(y_i)=2ny_i$ for all $i=1,2,...,n+1$ and

$$\prod_{i=1}^{n+1}(tx_i+y_i)=\sum_{i=0}^{n+1}p_it^{n+1-i}$$

for any real t, prove, that for all $k=1,...,n+1$

$$f(p_{k-1})=kp_k+(n+1)(n+k-2)p_{k-1}$$

Day 2

1 In triangle ABC, $M \in (BC)$, $\frac{BM}{BC} = \alpha$, $N \in (CA)$, $\frac{CN}{CA} = \beta$, $P \in (AB)$, $\frac{AP}{AB} = \gamma$.
Let $AM \cap BN = \{D\}$, $BN \cap CP = \{E\}$, $CP \cap AM = \{F\}$. Prove that $S_{DEF} = S_{BMD} + S_{CNE} + S_{APF}$ iff $\alpha + \beta + \gamma = 1$.

2 Let $m \in N$ and $E(x,y,m) = (\frac{72}{x})^m + (\frac{72}{y})^m - x^m - y^m$, where x and y are positive divisors of 72.
a) Prove that there exist infinitely many natural numbers m so, that 2005 divides $E(3,12,m)$ and $E(9,6,m)$.
b) Find the smallest positive integer number m_0 so, that 2005 divides $E(3,12,m_0)$ and $E(9,6,m_0)$.

3 Does there exist such a configuration of 22 circles and 22 point, that any circle contains at leats 7 points and any point belongs at least to 7 circles?

4 Find the largest positive p ($p>1$) such, that $\forall a,b,c \in [\frac{1}{p}, p]$ the following inequality takes place

$$9(ab+bc+ca)(a^2+b^2+c^2) \geq (a+b+c)^4$$

Day 3

1 Let ABC and $A_1B_1C_1$ be two triangles. Prove that
$\frac{a}{a_1} + \frac{b}{b_1} + \frac{c}{c_1} \leq \frac{3R}{2r_1}$,

where $a = BC$, $b = CA$, $c = AB$ are the sidelengths of triangle ABC, where $a_1 = B_1C_1$, $b_1 = C_1A_1$, $c_1 = A_1B_1$ are the sidelengths of triangle $A_1B_1C_1$, where R is the circumradius of triangle ABC and r_1 is the inradius of triangle $A_1B_1C_1$.

2 Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcenters of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extensions points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

Proposed by Hojoo Lee, Korea

3 For an $n \times n$ matrix A, let X_i be the set of entries in row i, and Y_j the set of entries in column j, $1 \leq i,j \leq n$. We say that A is *golden* if $X_1, \dots, X_n, Y_1, \dots, Y_n$ are distinct sets. Find the least integer n such that there exists a 2004×2004 golden matrix with entries in the set $\{1, 2, \dots, n\}$.

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4 Given functions $f, g : N^* \to N^*$, g is surjective and $2f(n)^2 = n^2 + g(n)^2, \forall n > 0$. Prove that if $|f(n) - n| \le 2005\sqrt{n}, \forall n > 0$, then $f(n) = n$ for infinitely many n.
