Moldova Team Selection Test 2006

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Day 1

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1 Determine all even numbers n, $n \in \mathbb{N}$ such that

$$\frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_k}=\frac{1620}{1003},$$

where $d_1, d_2, \ldots, d_k$ are all different divisors of n.

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2 Consider a right-angled triangle ABC with the hypothenuse $AB=1$. The bisector of $\angle ACB$ cuts the medians BE and AF at P and M, respectively. If $AF \cap BE = \{P\}$, determine the maximum value of the area of $\triangle MNP$.

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3 Let a, b, c be sides of the triangle. Prove that

$$a^2\left(\frac{b}{c}-1\right)+b^2\left(\frac{c}{a}-1\right)+c^2\left(\frac{a}{b}-1\right)\geq 0.$$

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4 Let m circles intersect in points A and B. We write numbers using the following algorithm: we write 1 in points A and B, in every midpoint of the open arc AB we write 2, then between every two numbers written in the midpoint we write their sum and so on repeating n times. Let
$r(n,m)$
be the number of appearances of the number n writing all of them on our m circles.

a) Determine $r(n,m)$;

b) For $n=2006$, find the smallest m for which $r(n,m)$ is a perfect square.

Example for half arc: $1-1$; $1-2-1$; $1-3-2-3-1$; $1-4-3-5-2-5-3-4-1$; $1-5-4-7-3-8-5-7-2-7-5-8-3-7-4-5-1$...

Day 2

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1 Let (a_n) be the Lucas sequence: $a_0=2, a_1=1, a_{n+1}=a_n+a_{n-1}$ for $n\geq 1$. Show that a_{59} divides $(a_{30})^{59}-1$.

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2 Let C_1 be a circle inside the circle C_2 and let P in the interior of C_1, Q in the exterior of C_2. One draws variable lines l_i through P, not passing through Q. Let l_i intersect C_1 in A_i, B_i, and let the circumcircle of QA_iB_i intersect C_2 in M_i, N_i. Show that all lines M_i, N_i are concurrent.

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3 Let a, b, c be sides of a triangle and p its semiperimeter. Show that $a\sqrt{\frac{(p-b)(p-c)}{bc}} + b\sqrt{\frac{(p-c)(p-a)}{ac}} + c\sqrt{\frac{(p-a)(p-b)}{ab}} \geq p$

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4 Let $A = \{1, 2, \dots, n\}$. Find the number of unordered triples (X, Y, Z) that satisfy $X \bigcup Y \bigcup Z = A$

Day 3 March 26th

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1 Let the point P in the interior of the triangle ABC. $(AP, (BP, (CP$ intersect the circumcircle of ABC at A_1, B_1, C_1. Prove that the maximal value of the sum of the areas A_1BC, B_1AC, C_1AB is $p(R - r)$, where p, r, R are the usual notations for the triangle ABC.

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2 Let $n \in N$ $n \geq 2$ and the set X with $n + 1$ elements. The ordered sequences $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots b_n)$ of distinct elements of X are said to be *separated* if there exists $i \neq j$ such that $a_i = b_j$. Determine the maximal number of ordered sequences of n elements from X such that any two of them are *separated*.
Note: ordered means that, for example $(1, 2, 3) \neq (2, 3, 1)$.

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3 Positive real numbers a, b, c satisfy the relation $abc = 1$. Prove the inequality: $\frac{a+3}{(a+1)^2} + \frac{b+3}{(b+1)^2} + \frac{c+3}{(c+1)^2} \geq 3$.

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4 Let $f(n)$ denote the number of permutations $(a_1, a_2, \dots, a_n)$ of the set $\{1, 2, \dots, n\}$, which satisfy the conditions: $a_1 = 1$ and $|a_i - a_{i+1}| \leq 2$, for any $i = 1, 2, \dots, n - 1$. Prove that $f(2006)$ is divisible by 3.
