Moldova Team Selection Test 2008

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Day 1

1 Let p be a prime number. Solve in $\mathbb{N}_0 \times \mathbb{N}_0$ the equation $x^3+y^3-3xy=p-1$.

2 We say the set $\{1,2,\dots,3k\}$ has property D if it can be partitioned into disjoint triples so that in each of them a number equals the sum of the other two.

(a) Prove that $\{1,2,\dots,3324\}$ has property D.

(b) Prove that $\{1,2,\dots,3309\}$ hasn't property D.

3 Let $\Gamma(I,r)$ and $\Gamma(O,R)$ denote the incircle and circumcircle, respectively, of a triangle ABC. Consider all the triangels $A_iB_iC_i$ which are simultaneously inscribed in $\Gamma(O,R)$ and circumscribed to $\Gamma(I,r)$. Prove that the centroids of these triangles are concyclic.

4 A non-zero polynomial $S \in \mathbb{R}[X,Y]$ is called homogeneous of degree d if there is a positive integer d so that $S(\lambda x, \lambda y)=\lambda^d S(x,y)$ for any $\lambda \in \mathbb{R}$. Let $P, Q \in \mathbb{R}[X,Y]$ so that Q is homogeneous and P divides Q (that is, $P|Q$). Prove that P is homogeneous too.

Day 2 March 29th

1 Find all solutions $(x,y) \in \mathbb{R}\times\mathbb{R}$ of the following system: $\begin{cases} x^3+3xy^2=49, \\ x^2+8xy+y^2=8y+17x. \end{cases}$

2 Let $a_1,\dots,a_n$ be positive reals so that $a_1+a_2+\dots+a_n \le \frac{n}{2}$. Find the minimal value of $\sqrt{a_1^2+\frac{1}{a_2^2}}+\sqrt{a_2^2+\frac{1}{a_3^2}}+\dots+\sqrt{a_n^2+\frac{1}{a_1^2}}$.

3 Let ω be the circumcircle of ABC and let D be a fixed point on BC, $D \neq B$, $D \neq C$. Let X be a variable point on (BC), $X \neq D$. Let Y be the second intersection point of AX and ω. Prove that the circumcircle of XYD passes through a fixed point.

4 Find the number of even permutations of $\{1,2,\dots,n\}$ with no fixed points.

Day 3 March 30th

1 Determine a subset $A\subset \mathbb{N}^*$ having 5 different elements, so that the sum of the squares of its elements equals their product.
Do not simply post the subset, show how you found it.

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2 Let p be a prime number and k,n positive integers so that $\gcd(p,n)=1$. Prove that $\binom{n\cdot p^k}{p^k}$ and p are coprime.

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3 In triangle ABC the bisector of $\angle ACB$ intersects AB at D. Consider an arbitrary circle O passing through C and D, so that it is not tangent to BC or CA. Let $O\cap BC=\{M\}$ and $O\cap CA=\{N\}$.
a) Prove that there is a circle S so that DM and DN are tangent to S in M and N, respectively.
b) Circle S intersects lines BC and CA in P and Q respectively. Prove that the lengths of MP and NQ do not depend on the choice of circle O.

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4 A non-empty set S of positive integers is said to be *good* if there is a coloring with 2008 colors of all positive integers so that no number in S is the sum of two different positive integers (not necessarily in S) of the same color. Find the largest value t can take so that the set $S=\{a+1,a+2,a+3,\ldots,a+t\}$ is good, for any positive integer a.

I have the feeling that I've seen this problem before, so if I'm right, maybe someone can post some links...
