Moldova Team Selection Test 2009

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Day 1

1 Let $m,n\in\mathbb{N}^*$. Find the least n for which exists m, such that rectangle $(3m+2)\times(4m+3)$ can be covered with $\dfrac{n(n+1)}{2}$ squares, among which exist n squares of length 1, $n-1$ of length 2, ..., 1 square of length n. For the found value of n give the example of covering.

2 Let $m,n\in\mathbb{N}$, $n\geq 2$ and numbers $a_i>0$, $i=\overline{1,n}$, such that $\sum a_i=1$. Prove that

$$\frac{a_1^{2-m}+a_2+...+a_{n-1}}{1-a_1}+\frac{a_2^{2-m}+a_3+...+a_n}{1-a_1}+...+\frac{a_n^{2-m}+a_1+...+a_{n-2}}{1-a_1}\geq n+\frac{n^m-n}{n-1}$$

3 Quadrilateral $ABCD$ is inscribed in the circle of diameter BD. Point A_1 is reflection of point A wrt BD and B_1 is reflection of B wrt AC. Denote $\{P\}=CA_1\cap BD$ and $\{Q\}=DB_1\cap AC$. Prove that $AC\perp PQ$.

4 Let p be a prime divisor of $n\geq 2$. Prove that there exists a set of natural numbers $A=\{a_1,a_2,...,a_n\}$ such that product of any two numbers from A is divisible by the sum of any p numbers from A.

Day 2

1 For any $m\in\mathbb{N}^*$ solve the ecuation

$$\left\{\left(x+\frac{1}{m}\right)^3\right\}=x^3$$

2 Determine all functions $f:[0;+\infty)\rightarrow[0;+\infty)$, such that

$$f(x+y-z)+f(2\sqrt{xz})+f(2\sqrt{yz})=f(x+y+z)$$

for all $x,y,z\in[0;+\infty)$, for which $x+y\geq z$.

3 A circle Ω_1 is tangent outwardly to the circle Ω_2 of bigger radius. Line t_1 is tangent at points A and D to the circles Ω_1 and Ω_2 respectively. Line t_2, parallel to t_1, is tangent to the circle Ω_1 and cuts Ω_2 at points E and F. Point C belongs to the circle Ω_2 such that D and C are separated by the line EF. Denote B the intersection of EF and CD. Prove that circumcircle of ABC is tangent to the line AD.

4 Let m and n be two nonzero natural numbers. In every cell 1×1 of the rectangular table $2m \times 2n$ are put signs $+$ or $-$. We call *cross* an union of all cells which are situated in a line and in a column of the table. Cell, which is situated at the intersection of these line and column is called *center of the cross*. A transformation is defined in the following way: firstly we mark all points with the sign $-$. Then consecutively, for every marked cell we change the signs in the cross, whose center is the choosen cell. We call a table *accesible* if it can be obtained from another table after one transformation.
Find the number of all *accesible* tables.

Day 3

1 Points X, Y and Z are situated on the sides (BC), (CA) and (AB) of the triangles ABC, such that triangles XYZ and ABC are similiar. Prove that circumcircle of AYZ passes through a fixed point.

2 Let M be a set of aritmetic progressions with integer terms and ratio bigger than 1.

a) Prove that the set of the integers $\mathbb{Z}$ can be written as union of the finite number of the progessions from M with different ratios.

b) Prove that the set of the integers $\mathbb{Z}$ can not be written as union of the finite number of the progessions from M with ratios integer numbers, any two of them coprime.

3 Weightlifter Ruslan has just finished the exercise with a weight, which has n small weights on one side and n on the another. At each stage he takes some weights from one of the sides, such that at any moment the difference of the numbers of weights on the sides does not exceed k. What is the minimal number of stages (in function if n and k), which Ruslan need to take off all weights..

4 let x, y, z be real number in the interval $[\frac{1}{2}; 2]$ and a, b, c a permutation of them. Prove the inequality:
$$\frac{60a^2-1}{4xy+5z}+\frac{60b^2-1}{4yz+5x}+\frac{60c^2-1}{4zx+5y}\geq 12$$

Day 4

1 Let $ABCD$ be a trapezoid with $AB \parallel CD$. Exterior equilateral triangles ABE and CDF are constructed. Prove that lines AC, BD and EF are concurrent.

2 $f(x)$ and $g(x)$ are two polynomials with nonzero degrees and integer coefficients, such that $g(x)$ is a divisor of $f(x)$ and the polynomial $f(x) + 2009$ has 50 integer roots. Prove that the degree of $g(x)$ is at least 5.

3 The sequence $(a_n)_{n\in\mathbb{N}}$ is defined as follows:

$$a_n = \frac{2}{3+1} + \frac{2^2}{3^2+1} + \frac{2^3}{3^4+1} + \ldots + \frac{2^{n+1}}{3^{2^n}+1}$$

Prove that $a_n < 1$ for any $n \in \mathbb{N}$

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4 Let X be a group of people, where any two people are friends or enemies. Each pair of friends from X doesn't have any common friends, and any two enemies have exactly two common friends. Prove that each person from X has the same number of friends as others.
