Moldova Team Selection Test 2013

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Day 1

1 For any positive real numbers x, y, z, prove that $\frac{x}{y}+\frac{y}{z}+\frac{z}{x} \geq \frac{z(x+y)}{y(y+z)}+\frac{x(z+y)}{z(x+z)}+\frac{y(x+z)}{x(x+y)}$

2 We call a triangle $\triangle ABC$, Q-angled if $\tan\angle A, \tan\angle B, \tan\angle C \in \mathbb{Q}$, where $\angle A, \angle B, \angle C$ are the interior angles of the triangle $\triangle ABC$. $a)$ Prove that Q-angled triangles exist; $b)$ Let triangle $\triangle ABC$ be Q-angled. Prove that for any non-negative integer n, numbers of the form $E_n = \sin^n\angle A \sin^n\angle B \sin^n\angle C + \cos^n\angle A \cos^n\angle B \cos^n\angle C$ are rational.

3 Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

4 Consider a positive real number a and a positive integer m. The sequence $(x_k)_{k\in\mathbb{Z}^+}$ is defined as: $x_1 = 1$, $x_2 = a$, $x_{n+2} = \sqrt[m+1]{x_{n+1}^m x_n}$. $a)$ Prove that the sequence is converging. $b)$ Find $\lim_{n\to\infty} x_n$.

Day 2

1 Let m be the number of ordered solutions (a, b, c, d, e) satisfying: $1)$ $a, b, c, d, e \in \mathbb{Z}^+$; $2)$ $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}=1$;
Prove that m is odd.

2 Let $a_n = 1 + n!(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+...+\frac{1}{n!})$ for any $n \in \mathbb{Z}^+$. Consider a_n points in the plane,no 3 of them collinear.The segments between any 2 of them are colored in one of n colors. Prove that among them there exist 3 points forming a monochromatic triangle.

3 The diagonals of a trapezoid $ABCD$ with $AD \parallel BC$ intersect at point P. Point Q lies between the parallel lines AD and BC such that the line CD separates points P and Q, and $\angle AQD = \angle CQB$. Prove that $\angle BQP = \angle DAQ$.

4 Prove that for any positive real numbers a_i, b_i, c_i with $i = 1, 2, 3$, $(a_1^3+b_1^3+c_1^3+1)(a_2^3+b_2^3+c_2^3+1)(a_3^3+b_3^3+c_3^3+1) \geq \frac{3}{4}(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)$

Day 3

1 Let $A = 20132013...2013$ be formed by joining 2013, 165 times. Prove that $2013^2 \mid A$.

2 Find all pairs of real numbers (x, y) satisfying $\begin{cases} 2x^2+xy &= 1 \\ \frac{9x^2}{2(1-x)^4} &= 1+\frac{3xy}{2(1-x)^2} \end{cases}$

3 Consider the obtuse-angled triangle $\triangle ABC$ and its side lengths a, b, c. Prove that $a^3\cos\angle A + b^3\cos\angle B + c^3\cos\angle C < abc$.

4 Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

Day 4

1 Consider real numbers x, y, z such that $x, y, z > 0$. Prove that

$$(xy+yz+xz)\left(\frac{1}{x^2+y^2}+\frac{1}{x^2+z^2}+\frac{1}{y^2+z^2}\right) > \frac{5}{2}.$$

2 Consider a board on 2013×2013 squares, what is the maximum number of chess knights that can be placed so that no 2 attack each other?

3 Consider the triangle $\triangle ABC$ with $AB \neq AC$. Let point O be the circumcenter of $\triangle ABC$. Let the angle bisector of $\angle BAC$ intersect BC at point D. Let E be the reflection of point D across the midpoint of the segment BC. The lines perpendicular to BC in points D, E intersect the lines AO, AD at the points X, Y respectively. Prove that the quadrilateral B, X, C, Y is cyclic.

4 p is a 4k+3 prime. Prove that there are infinite p which satisfies $p|2^ny+1$. y is an random integer.