Moldova National Olympiad 2002

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– Grade level 7

Day 1

1 Before going to vacation, each of the 7 pupils decided to send to each of the 3 classmates one postcard. Is it possible that each student receives postcards only from the classmates he has sent postcards?

2 Let $a, b, c \geq 0$ such that $a+b+c=1$. Prove that:
$a^2+b^2+c^2 \geq 4(ab+bc+ca)-1$

3 Consider an angle $\angle DEF$, and the fixed points B and C on the semiline $(EF$ and the variable point A on $(ED$. Determine the position of A on $(ED$ such that the sum $AB+AC$ is minimum.

4 Twelve teams participated in a soccer tournament. According to the rules, one team gets 2 points for a victory, 1 point for a draw and 0 points for a defeat. When the tournament was over, all teams had distinct numbers of points, and the team ranked second had as many points as the teams ranked on the last five places in total. Who won the match between the fourth and the eighth place teams?

Day 2

1 Volume A equals one fourth of the sum of the volumes B and C, while volume B equals one sixth of the sum of the volumes C and A.
Find the ratio of the volume C to the sum of the volumes A and B.

2 Five parcels of land are given. In each step, we divide one parcel into three or four smaller ones. Assume that, after several steps, the number of obtained parcels equals four times the number of steps made. How many steps were performed?

3 In a triangle ABC, the angle bisector at B intersects AC at D and the circumcircle again at E. The circumcircle of the triangle DAE meets the segment AB again at F. Prove that the triangles DBC and DBF are congruent.

4 Prove that there are infinitely many triplets (a, b, c) that satisfy the following equalities:
$\dfrac{2a-b+6}{4a+c+2}=\dfrac{b-2c}{a-c}=\dfrac{2a+b+2c-2}{6a+2c-2}$

– Grade level 8

Day 1

1 Find all real solutions of the equation: $[x]+[x+\dfrac{1}{2}]+[x+\dfrac{2}{3}]=2002$

2 Given a positive integer k, there is a positive integer n with the property that one can obtain the sum of the first n positive integers by writing some k digits to the right of n. Find the remainder of n when dividing at 9.

3 Consider a circle $\Gamma(O,R)$ and a point P found in the interior of this circle. Consider a chord AB of Γ that passes through P. Suppose that the tangents to Γ at the points A and B intersect at Q. Let $M\in QA$ and $N\in QB$ s.t. $PM\perp QA$ and $PN\perp QB$. Prove that the value of $\frac{1}{PN}+\frac{1}{PM}$ doesn't depend of choosing the chord AB.

4 All the internal phone numbers in a certain company have four digits. The director wants the phone numbers of the administration offices to consist of digits 1, 2, 3 only, and that any of these phone numbers coincide in at most one position. What is the maximum number of distinct phone numbers that these offices can have ?

Day 2

1 Several pupils wrote a solution of a math problem on the blackboard on the break. When the teacher came in, a pupil was just clearing the blackboard, so the teacher could only observe that there was a rectangle with the sides of integer lenghts and a diagonal of lenght 2002. Then the teacher pointed out that there was a computation error in pupils' solution. Why did he conclude that?

2 From a set of consecutive natural numbers one number is excluded so that the aritmetic mean of the remaining numbers is 50.55. Find the initial set of numbers and the excluded number.

3 In a triangle ABC, the bisectors of the angles at B and C meet the opposite sides B_1 and C_1, respectively. Let T be the midpoint AB_1. Lines BT and B_1C_1 meet at E and lines AB and CE meet at L. Prove that the lines TL and B_1C_1 have a point in common.

4 Let $x\in\mathbb{R}$. Find the minimum and maximum values of the expresion:
$E=\dfrac{(1+x)^8+16x^4}{(1+x^2)^4}$

– Grade level 9

Day 1

1 Consider the real numbers $a \neq 0, b, c$ such that the function $f(x) = ax^2 + bx + c$ satisfies $|f(x)| \leq 1$ for all $x \in [0, 1]$. Find the greatest possible value of $|a| + |b| + |c|$.

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2 Does there exist a positive integer $n > 1$ such that n is a power of 2 and one of the numbers obtained by permuting its (decimal) digits is a power of 3 ?

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3 Prove that for any $n \in \mathbb{N}$ the number $1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n+1}$ is not an integer.

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4 Let $ABCD$ be a convex quadrilateral and let N on side AD and M on side BC be points such that $\frac{AN}{ND} = \frac{BM}{MC}$. The lines AM and BN intersect at P, while the lines CN and DM intersect at Q. Prove that if $S_{ABP} + S_{CDQ} = S_{MNPQ}$, then either $AD \parallel BC$ or N is the midpoint of DA.

Day 2

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1 Integers $a_1, a_2, \ldots a_9$ satisfy the relations $a_{k+1} = a_k^3 + a_k^2 + a_k + 2$ for $k = 1, 2, ..., 8$. Prove that among these numbers there exist three with a common divisor greater than 1.

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2 The coefficients of the equation $ax^2 + bx + c = 0$, where $a \neq 0$, satisfy the inequality $(a + b + c)(4a - 2b + c) < 0$. Prove that this equation has 2 real distinct solutions.

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3 Let $a, b, c > 0$. Prove that:
$\frac{a}{2a+b} + \frac{b}{2b+c} + \frac{c}{2c+a} \leq 1$

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4 The circles C_1 and C_2 with centers O_1 and O_2 respectively are externally tangent. Their common tangent not intersecting the segment O_1O_2 touches C_1 at A and C_2 at B. Let C be the reflection of A in O_1O_2 and P be the intersection of AC and O_1O_2. Line BP meets C_2 again at L. Prove that line CL is tangent to the circle C_2.

– Grade level 10

Day 1

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1 We are given three nuggets of weights 1 kg, 2 kg and 3 kg, containing different percentages of gold, and need to cut each nugget into two parts so that the obtained parts can be alloyed into two ingots of weights 1 kg ande 5 kg containing the same proportion of gold. How we can do that?

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2 Let $n \geq 3$ distinct non-collinear points be given on a plane. Show that there is a closed simple polygonal line passing through each point.

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3 The sides AB,BC and CA of the triangle ABC are tangent to the incircle of the triangle ABC with center I at the points C_1,A_1 and B_1, respectively.Let B_2 be the midpoint of the side AC.Prove that the lines B_1I, A_1C_1 and BB_2 are concurrent.

4 In each line and column of a table $(2n+1)\times(2n+1)$ are written arbitrarly the numbers $1, 2, ..., 2n+1$. It was constated that the repartition of the numbers is symmetric to the main diagonal of this table. Prove that all the numbers on the main diagonal are distinct.

Day 2

1 Find all triplets of primes in the form $(p, 2p+1, 4p+1)$.

2 Let $a, b, c \in \mathbb{R}$ such that $a \geq b \geq c > 1$. Prove the inequality:
$\log_c \log_c b + \log_b \log_b a + \log_a \log_a c \geq 0$

3 There are 16 persons in a company, each of which likes exactly 8 other persons. Show that there exist two persons who like each other.

4 Let the triangle ADB_1 s.t. $m(\angle DAB_1) \neq 90^\circ$.On the sides of this triangle externally are constructed the squares $ABCD$
and $AB_1C_1D_1$ with centers O_1 and O_2, respectively.Prove that the circumcircles of the triangles BAB_1, DAD_1 and O_1AO_2 share a common point, that differs from A.

– Grade level 11

Day 1

1 The sequence (a_n) is defined by $a_1 \in (0,1)$ and $a_{n+1} = a_n(1-a_n)$ for $n \geq 1$.
Prove that $\lim_{n\to\infty} na_n = 1$

2 For every nonnegative integer n and every real number x prove the inequality:
$|\cos x| + |\cos 2x| + \ldots + |\cos 2^n x| \geq \dfrac{n}{2\sqrt{2}}$

3 Let P be a polyhedron whose all edges are congruent and tangent to a sphere. Suppose that one of the facesof P has an odd number of sides. Prove that all vertices of P lie on a single sphere.

4 At least two of the nonnegative real numbers $a_1, a_2, ..., a_n$ aer nonzero. Decide whether a or b is larger if
$a = \sqrt[2002]{a_1^{2002} + a_2^{2002} + \ldots + a_n^{2002}}$

and
$b = \sqrt[2003]{a_1^{2003} + a_2^{2003} + \ldots + a_n^{2003}}$

Day 2

1 Solve in $\mathbb{R}$ the equation $\sqrt{1-x} = 2x^2 - 1 + 2x\sqrt{1-x^2}$.

2 Can a square of side 1024 be partitioned into 31 squares?Can a square of side 1023 be partitioned into 30 squares, one of which has a s side lenght not exceeding 1?

3 Let $a, b > 0$ such that $a \neq b$. Prove that:
$\sqrt{ab} < \dfrac{a-b}{\ln a - \ln b} < \dfrac{a+b}{2}$

4 The circumradius of a tetrahedron $ABCD$ is R, and the lenghts of the segments connecting the vertices A, B, C, D with the centroids of the opposite faces are equal to m_a, m_b, m_c and m_d, respectively. Prove that:
$m_a + m_b + m_c + m_d \leq \dfrac{16}{3}R$

– Grade level 12

12.5 Let $0 \leq a \leq b \leq c \leq 3$
Prove : $(a-b)(a^2-9)+(a-c)(b^2-9)+(b-c)(c^2-9) \leq 36$

12.6 Let A,B,C be three collinear points and a circle T(A,r).
If M and N are two diametrical opposite variable points on T,
Find locus geometrical of the intersection BM and CN.

12.8

$$\mathbf{\sum_{n=1}^{\infty} 3^n . \sin^3 (\frac{\pi}{3^n})} = ?$$