Moldova National Olympiad 2006

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–	Grade 8
8.4	Sum of 100 natural distinct numbers is 9999. Prove that 2006 divide their product.
–	Grade 9
9.1	Let a, b, c be positive real numbers such that $a+b+c=2005$. Find the minimum value of the expression: $$E=a^{2006}+b^{2006}+c^{2006}+\frac{(ab)^{2004}+(bc)^{2004}+(ca)^{2004}}{(abc)^{2004}}$$
–	Grade 10
10.1	Let a, b be the smaller sides of a right triangle. Let c be the hypothenuse and h be the altitude from the right angle. Fint the maximal value of $\frac{c+h}{a+b}$.
10.2	Let n be a positive integer, $n\geq 2$. Let $M=\{0,1,2,\dots n-1\}$. For an integer nonzero number a we define the function $f_a: M\longrightarrow M$, such that $f_a(x)$ is the remainder when dividing ax at n. Find a necessary and sufficient condition such that f_a is bijective. And if f_a is bijective and n is a prime number, prove that $a^{n(n-1)}-1$ is divisible by n^2.
10.3	A convex quadrilateral $ABCD$ is inscribed in a circle. The tangents to the circle through A and C intersect at a point P, such that this point P does not lie on BD, and such that $PA^2=PB\cdot PD$. Prove that the line BD passes through the midpoint of AC.
10.4	Find all real values of the real parameter a such that the equation $$2x^2-6ax+4a^2-2a-2+\log_2(2x^2+2x-6ax+4a^2)=$$ $$=\log_2(x^2+2x-3ax+2a^2+a+1).$$ has a unique solution.
10.5	Let x_1, x_2, ..., x_n be n real numbers in $\left(\frac{1}{4},\frac{2}{3}\right)$. Find the minimal value of the expression: $$\log_{\frac{3}{2}x_1}\left(\frac{1}{2}-\frac{1}{36x_2^2}\right)+\log_{\frac{3}{2}x_2}\left(\frac{1}{2}-\frac{1}{36x_3^2}\right)+\dots+\log_{\frac{3}{2}x_n}\left(\frac{1}{2}-\frac{1}{36x_1^2}\right).$$

10.6 Let a triangle ABC satisfy $AC = BC$; in other words, let ABC be an isosceles triangle with base AB. Let P be a point inside the triangle ABC such that $\angle PAB = \angle PBC$. Denote by M the midpoint of the segment AB. Show that $\angle APM + \angle BPC = 180°$.

10.7 Consider an octogon with equal angles and rational side lengths. Prove that it has a symmetry center.

10.8 Let $M = \{x^2+x \mid x \in \mathbb{N}^\star\}$. Prove that for every integer $k \geq 2$ there exist elements $a_1, a_2, \dots, a_k, b_k$ from M, such that $a_1 + a_2 + \dots + a_k = b_k$.

– Grade 11

11.1 Let $n \in \mathbb{N}^*$. Prove that

$$\lim_{x\to 0} \frac{(1+x^2)^{n+1} - \prod_{k=1}^{n} \cos kx}{x \sum_{k=1}^{n} \sin kx} = \frac{2n^2+n+12}{6n}.$$

11.2 Function $f : [a, b] \to \mathbb{R}$, $0 < a < b$ is continuous on $[a, b]$ and differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$f'(c) = \frac{1}{a-c} + \frac{1}{b-c} + \frac{1}{a+b}.$$

11.3 Let $ABCDE$ be a right quadrangular pyramid with vertex E and height EO. Point S divides this height in the ratio $ES : SO = m$. In which ratio does the plane (ABC) divide the lateral area of the pyramid.

11.4 On each of the 2006 cards a natural number is written. Cards are placed arbitrarily in a row. 2 players take in turns a card from any end of the row until all the cards are taken. After that each player calculates sum of the numbers written of his cards. If the sum of the first player is not less then the sum of the second one then the first player wins. Show that there's a winning strategy for the first player.

11.5 Let $n \in \mathbb{N}^*$. Solve the equation $\sum_{k=0}^{n} C_n^k \cos 2kx = \cos nx$ in $\mathbb{R}$.

11.6 Sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ satisfy the relations $x_n = 4x_{n-1} + 3y_{n-1}$ and $y_n = 2x_{n-1} + 3y_{n-1}$ for $n \geq 1$. If $x_1 = y_1 = 5$ find x_n and y_n.

Calculate $\lim_{n\to\infty} \frac{x_n}{y_n}$.

11.7 Let $n \in \mathbb{N}^*$. $2n+3$ points on the plane are given so that no 3 lie on a line and no 4 lie on a circle. Is it possible to find 3 points so that the interior of the circle passing through them would contain exactly n of the remaining points.

11.8 Given an alfabet of n letters. A sequence of letters such that between any 2 identical letters there are no 2 identical letters is called a *word*.

a) Find the maximal possible length of a *word*.

b) Find the number of the *words* of maximal length.

– Grade 12

12.2 Let $a, b, n \in \mathbb{N}$, with $a, b \geq 2$. Also, let $I_1(n) = \int_0^1 \lfloor a^n x \rfloor \, dx$ and $I_2(n) = \int_0^1 \lfloor b^n x \rfloor \, dx$. Find $\lim_{n\to\infty} \dfrac{I_1(n)}{I_2(n)}$.

12.4 Let $P(x) = x^n + a_1 x^{n-1} + ... + a_{n-1} x + (-1)^n$, $a_i \in C$, $n \geq 2$ with all roots having same modulo. Prove that $P(-1) \in R$

12.5 Let $a_1, a_2, ..., a_n$ be real positive numbers and $k > m, k, m$ natural numbers. Prove that $(n-1)(a_1^m + a_2^m + ... + a_n^m) \leq \frac{a_2^k + a_3^k + ... + a_n^k}{a_1^{k-m}} + \frac{a_1^k + a_3^k + ... + a_n^k}{a_2^{k-m}} + ... + \frac{a_1^k + a_2^k + ... + a_{n-1}^k}{a_n^{k-m}}$
