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– Paper 1

1 Three points X, Y and Z are given that are, respectively, the circumcenter of a triangle ABC, the mid-point of BC, and the foot of the altitude from B on AC. Show how to reconstruct the triangle ABC.

2 Problem:
Find all polynomials satisfying the equation $f(x^2) = (f(x))^2$
for all real numbers x.
I'm not exactly sure where to start though it doesn't look too difficult. Thanks!

3 Three operations f, g and h are defined on subsets of the natural numbers $\mathbb{N}$ as follows: $f(n) = 10n$, if n is a positive integer; $g(n) = 10n + 4$, if n is a positive integer; $h(n) = \frac{n}{2}$, if n is an *even* positive integer.
Prove that, starting from 4, every natural number can be constructed by performing a finite number of operations f, g and h in some order.

[For example: $35 = h(f(h(g(h(h(4))))))$.]

4 Eight politicians stranded on a desert island on January 1st, 1991, decided to establish a parliament.
They decided on the following rules of attendance:

(a) There should always be at least one person present on each day.

(b) On no two days should the same subset attend.

(c) The members present on day N should include for each $K < N$, $(K \geq 1)$ at least one member who was present on day K.

For how many days can the parliament sit before one of the rules is broken?

5 Find all polynomials

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n$$

with the following properties

(a) all the coefficients $a_1, a_2, ..., a_n$ belong to the set $\{-1, 1\}$; and
(b) all the roots of the equation $f(x) = 0$ are real.

– Paper 2

1 Problem. The sum of two consecutive squares can be a square; for instance $3^2 + 4^2 = 5^2$.

(a) Prove that the sum of m consecutive squares cannot be a square for $m \in \{3, 4, 5, 6\}$.
(b) Find an example of eleven consecutive squares whose sum is a square.

Can anyone help me with this?
Thanks.

2 Let

$$a_n = \frac{n^2+1}{\sqrt{n^4+4}}, \quad n = 1, 2, 3, \ldots$$

and let b_n be the product of $a_1, a_2, a_3, \ldots, a_n$. Prove that

$$\frac{b_n}{\sqrt{2}} = \frac{\sqrt{n^2+1}}{\sqrt{n^2+2n+2}},$$

and deduce that

$$\frac{1}{n^3+1} < \frac{b_n}{\sqrt{2}} - \frac{n}{n+1} < \frac{1}{n^3}$$

for all positive integers n.

3 Let ABC be a triangle, and let the angle bisectors of its angles CAB and ABC meet the sides BC and CA at the points D and F, respectively. The lines AD and BF meet the line through the point C parallel to AB at the points E and G respectively, and we have $FG = DE$. Prove that $CA = CB$.

Original formulation:

Let ABC be a triangle and L the line through C parallel to the side AB. Let the internal bisector of the angle at A meet the side BC at D and the line L at E and let the internal bisector of the angle at B meet the side AC at F and the line L at G. If $GF = DE$, prove that $AC = BC$.

4 Let $\mathbb{P}$ be the set of positive rational numbers and let $f : \mathbb{P} \to \mathbb{P}$ be such that

$$f(x) + f\left(\frac{1}{x}\right) = 1$$

and

$$f(2x) = 2f(f(x))$$

for all $x \in \mathbb{P}$.
Find, with proof, an explicit expression for $f(x)$ for all $x \in \mathbb{P}$.

5 Let $\mathbb{Q}$ denote the set of rational numbers. A nonempty subset S of $\mathbb{Q}$ has the following properties:

(a) 0 is not in S;

(b) for each s_1, s_2 in S, the rational number s_1/s_2 is in S;

(c) there exists a nonzero number $q \in \mathbb{Q}\backslash S$ that has the property that every nonzero number in $\mathbb{Q}\backslash S$ is of the form qs for some s in S.

Prove that if x belongs to S, then there exists elements y, z in S such that $x = y + z$.