Nordic 2013

www.artofproblemsolving.com/community/c532728

by parmenides51

1 Let $(a_n)_{n\geq 1}$ be a sequence with $a_1 = 1$ and $a_{n+1} = \lfloor a_n + \sqrt{a_n} + \frac{1}{2} \rfloor$ for all $n \geq 1$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Find all $n \leq 2013$ such that a_n is a perfect square

---

2 In a football tournament there are n teams, with $n \geq 4$, and each pair of teams meets exactly once. Suppose that, at the end of the tournament, the final scores form an arithmetic sequence where each team scores 1 more point than the following team on the scoreboard. Determine the maximum possible score of the lowest scoring team, assuming usual scoring for football games (where the winner of a game gets 3 points, the loser 0 points, and if there is a tie both teams get 1 point).

---

3 Define a sequence $(n_k)_{k\geq 0}$ by $n_0 = n_1 = 1$, and $n_{2k} = n_k + n_{k-1}$ and $n_{2k+1} = n_k$ for $k \geq 1$. Let further $q_k = n_k$ / n_{k-1} for each $k \geq 1$. Show that every positive rational number is present exactly once in the sequence $(q_k)_{k\geq 1}$

---

4 Let ABC be an acute angled triangle, and H a point in its interior. Let the reflections of H through the sides AB and AC be called H_c and H_b , respectively, and let the reflections of H through the midpoints of these same sidesbe called $H_c^{'}$ and $H_b^{'}$, respectively. Show that the four points $H_b, H_b^{'}, H_c$, and $H_c^{'}$ are concyclic if and only if at least two of them coincide or H lies on the altitude from A in triangle ABC.
