www.artofproblemsolving.com/community/c533014
by laegolas, parmenides51, Hydrogen-Helium, AndrewTom

– Paper 1

1 Describe in geometric terms the set of points (x, y) in the plane such that x and y satisfy the condition $t^2 + yt + x \geq 0$ for all t with $-1 \leq t \leq 1$.

2 How many ordered triples (x, y, z) of real numbers satisfy the system of equations

$$x^2 + y^2 + z^2 = 9,$$

$$x^4 + y^4 + z^4 = 33,$$

$$xyz = -4?$$

3 Let A be a nonempty set with n elements. Find the number of ways of choosing a pair of subsets (B, C) of A such that B is a nonempty subset of C.

4 In a triangle ABC, the points A', B' and C' on the sides opposite A ,B and C, respectively, are such that the lines AA', BB' and CC' are concurrent. Prove that the diameter of the circumscribed circle of the triangle ABC equals the product $|AB'| \cdot |BC'| \cdot |CA'|$ divided by the area of the triangle $A'B'C'$.

5 Let ABC be a triangle such that the coordinates of the points A and B are rational numbers. Prove that the coordinates of C are rational if, and only if, $\tan A$, $\tan B$, and $\tan C$, when defined, are all rational numbers.

– Paper 2

1 Let $n > 2$ be an integer and let $m = \sum k^3$, where the sum is taken over all integers k with $1 \leq k < n$ that are relatively prime to n. Prove that n divides m.

2 If a_1 is a positive integer, form the sequence $a_1, a_2, a_3, \ldots$ by letting a_2 be the product of the digits of a_1, etc.. If a_k consists of a single digit, for some $k \geq 1$, a_k is called a *digital root* of a_1. It is easy to check that every positive integer has a unique root. (For example, if $a_1 = 24378$, then $a_2 = 1344$, $a_3 = 48$, $a_4 = 32$, $a_5 = 6$, and thus 6 is the digital root of 24378.) Prove that the digital root of a positive integer n equals 1 if, and only if, all the digits of n equal 1.

3 Let a, b, c and d be real numbers with $a \neq 0$. Prove that if all the roots of the cubic equation

$az^3 + bz^2 + cz + d = 0$

lie to the left of the imaginary axis in the complex plane, then

$ab > 0, bc - ad > 0, ad > 0.$

---

4 A convex pentagon has the property that each of its diagonals cuts off a triangle of unit area. Find the area of the pentagon.

---

5 If, for $k = 1, 2, \dots, n$, a_k and b_k are positive real numbers, prove that

$$\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n} \leq \sqrt[n]{(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)};$$

and that equality holds if, and only if,

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$
