Korea National Olympiad 2004

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1	For arbitrary real number x, the function $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(f(x))-x^2+x+3=0$. Show that the function f does not exist.
2	x and y are positive and relatively prime and z is an integer. They satisfy $(5z-4x)(5z-4y)=25xy$. Show that at least one of $10z+x+y$ or quotient of this number divided by 3 is a square number (i.e. prove that $10z+x+y$ or integer part of $\frac{10z+x+y}{3}$ is a square number).
3	Positive real numbers, $a_1,..,a_6$ satisfy $a_1^2+..+a_6^2=2$. Think six squares that has side length of a_i ($i=1,2,\dots,6$). Show that the squares can be packed inside a square of length 2, without overlapping.
4	Let k and N be positive real numbers which satisfy $k\le N$. For $1\le i\le k$, there are subsets A_i of $\{1,2,3,\dots,N\}$ that satisfy the following property. For arbitrary subset of $\{i_1,i_2,\dots,i_s\}\subset\{1,2,3,\dots,k\}$, $A_{i_1}\triangle A_{i_2}\triangle...\triangle A_{i_s}$ is not an empty set. Show that a subset $\{j_1,j_2,..,j_t\}\subset\{1,2,...,k\}$ exist that satisfies $n(A_{j_1}\triangle A_{j_2}\triangle\cdots\triangle A_{j_t})\ge k$. ($A\triangle B=A\cup B-A\cap B$)
5	A,B,C, and D are the four different points on the circle O in the order. Let the centre of the scribed circle of triangle ABC, which is tangent to BC, be O_1. Let the centre of the scribed circle of triangle ACD, which is tangent to CD, be O_2. (1) Show that the circumcentre of triangle ABO_1 is on the circle O. (2) Show that the circumcircle of triangle CO_1O_2 always pass through a fixed point on the circle O, when C is moving along arc BD.