Korea National Olympiad 2005

www.artofproblemsolving.com/community/c5339

by lightrhee

Day 1

1 For two positive integers a and b, which are relatively prime, find all integer that can be the great common divisor of $a+b$ and $\frac{a^{2005}+b^{2005}}{a+b}$.

2 For triangle ABC, P and Q satisfy $\angle BPA + \angle AQC = 90^\circ$. It is provided that the vertices of the triangle BAP and ACQ are ordered counterclockwise(or clockwise). Let the intersection of the circumcircles of the two triangles be N ($A \neq N$, however if A is the only intersection $A = N$), and the midpoint of segment BC be M. Show that the length of MN does not depend on P and Q.

3 For a positive integer K, define a sequence, $\{a_n\}_n$, as following $a_1 = K$,

$$a_{n+1} = \left\{ \begin{array}{ll} a_n - 1, & \text{if } a_n \text{ is even} \\ \frac{a_n-1}{2}, & \text{if } a_n \text{ is odd} \end{array} \right. ,$$

for all $n \geq 1$.

Find the smallest value of K, which makes a_{2005} the first term equal to 0.

4 Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, $f(x) \geq 0$ and for all real numbers x and y,

$$f(x+y) + f(x-y) - 2f(x) - 2y^2 = 0.$$

Day 2

5 Let P be a point that lies outside of circle O. A line passes through P and meets the circle at A and B, and another line passes through P and meets the circle at C and D. The point A is between P and B, C is between P and D. Let the intersection of segment AD and BC be L and construct E on ray $(PA$ so that $BL \cdot PE = DL \cdot PD$.

Show that M is the midpoint of the segment DE, where M is the intersection of lines PL and DE.

6 Real numbers $x_1, x_2, x_3, \cdots, x_n$ satisfy $x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2 = 1$. Show that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+x_2^2+x_3^2+\cdots+x_n^2} < \sqrt{\frac{n}{2}}.$$

7 For a positive integer n, let $f(n)$ be the number of factors of n^2+n+1. Show that there are infinitely many integers n which satisfy $f(n) \geq f(n+1)$.

8 A group of 6 students decided to make *study groups* and *service activity groups* according to the following principle:

Each group must have exactly 3 members. For any pair of students, there are same number of study groups and service activity groups that both of the students are members.

Supposing there are at least one group and no three students belong to the same study group and service activity group, find the minimum number of groups.