Korea National Olympiad 2007

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Day 1

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1 Consider the string of length 6 composed of three characters a, b, c. For each string, if two as are next to each other, or two bs are next to each other, then replace aa by b, and replace bb by a. Also, if a and b are next to each other, or two cs are next to each other, remove all two of them (i.e. delete ab, ba, cc). Determine the number of strings that can be reduced to c, the string of length 1, by the reducing processes mentioned above.

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2 $A_1B_1B_2A_2$ is a convex quadrilateral, and $A_1B_1 \neq A_2B_2$. Show that there exists a point M such that

$$\frac{A_1B_1}{A_2B_2} = \frac{MA_1}{MA_2} = \frac{MB_1}{MB_2}$$

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3 Let S be the set of all positive integers whose all digits are 1 or 2. Denote T_n as the set of all integers which is divisible by n, then find all positive integers n such that $S \cap T_n$ is an infinite set.

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4 Two real sequence $\{x_n\}$ and $\{y_n\}$ satisfies following recurrence formula;
$x_0 = 1, y_0 = 2007$ $x_{n+1} = x_n - (x_ny_n + x_{n+1}y_{n+1} - 2)(y_n + y_{n+1})$, $y_{n+1} = y_n - (x_ny_n + x_{n+1}y_{n+1} - 2)(x_n + x_{n+1})$

Then show that for all nonnegative integer n, ${x_n}^2 \leq 2007$.

Day 2

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1 For all positive reals a, b, and c, what is the value of positive constant k satisfies the following inequality? $\frac{a}{c+kb} + \frac{b}{a+kc} + \frac{c}{b+ka} \geq \frac{1}{2007}$.

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2 ABC is a triangle which is not isosceles. Let the circumcenter and orthocenter of ABC be O, H, respectively,
and the altitudes of ABC be AD, BC, CF. Let $K \neq A$ be the intersection of AD and circumcircle of triangle ABC, L be the intersection of OK and BC, M be the midpoint of BC, P be the intersection of AM and the line that passes L and perpendicular to BC, Q be the intersection of AD and the line that passes P and parallel to MH, R be the intersection of line EQ and AB, S be the intersection of FD and BE.
If $OL = KL$, then prove that two lines OH and RS are orthogonal.

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3 In each 2007^2 unit squares on chess board whose size is 2007×2007,
there lies one coin each square such that their "heads" face upward.
Consider the process that flips four consecutive coins on the same row, or flips four consecutive coins on the same column.
Doing this process finite times, we want to make the "tails" of all of coins face upward, except one that lies in the ith row and jth column.
Show that this is possible if and only if both of i and j are divisible by 4.

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4 For all positive integer $n \geq 2$, prove that product of all prime numbers less or equal than n is smaller than 4^n.
