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– Paper 1

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1 The real numbers α, β satisfy the equations

$$\alpha^3 - 3\alpha^2 + 5\alpha - 17 = 0$$

$$\beta^3 - 3\beta^2 + 5\beta + 11 = 0$$

Find $\alpha + \beta$.

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2 A positive integer n is called $good$ if it can be uniquely written simultaneously as $a_1 + a_2 + ... + a_k$ and as $a_1 a_2 ... a_k$, where a_i are positive integers and $k \geq 2$. (For example, 10 is good because $10 = 5 + 2 + 1 + 1 + 1 = 5 \cdot 2 \cdot 1 \cdot 1 \cdot 1$ is a unique expression of this form). Find, in terms of prime numbers, all good natural numbers.

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3 A line l is tangent to a circle S at A. For any points B, C on l on opposite sides of A, let the other tangents from B and C to S intersect at a point P. If B, C vary on l so that the product $AB \cdot AC$ is constant, find the locus of P.

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4 Let $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_0$ $(n \geq 1)$ be a polynomial with real coefficients such that $|f(0)| = f(1)$ and each root α of f is real and lies in the interval $[0, 1]$. Prove that the product of the roots does not exceed $\frac{1}{2^n}$.

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5 For a complex number $z = x + iy$ we denote by $P(z)$ the corresponding point (x, y) in the plane. Suppose $z_1, z_2, z_3, z_4, z_5, \alpha$ are nonzero complex numbers such that: (i) $P(z_1), ..., P(z_5)$ are vertices of a complex pentagon Q containing the origin O in its interior, and (ii) $P(\alpha z_1), ..., P(\alpha z_5)$ are all inside Q.
If $\alpha = p + iq$ $(p, q \in \mathbb{R})$, prove that $p^2 + q^2 \leq 1$ and $p + q \tan \frac{\pi}{5} \leq 1$.

– Paper 2

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1 Show that among any five points $P_1, ..., P_5$ with integer coordinates in the plane, there exists at least one pair (P_i, P_j), with $i \neq j$ such that the segment $P_i P_j$ contains a point Q with integer coordinates other than P_i, P_j.

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2 Let a_i, b_i $(i = 1, 2, ..., n)$ be real numbers such that the a_i are distinct, and suppose that there is a real number α such that the product $(a_i + b_1)(a_i + b_2)...(a_i + b_n)$ is equal to α for each i.

Prove that there is a real number β such that $(a_1+b_j)(a_2+b_j)...(a_n+b_j)$ is equal to β for each j.

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3 If $1 \le r \le n$ are integers, prove the identity:
$\sum_{d=1}^{\infty}\binom{n-r+1}{d}\binom{r-1}{d-1}=\binom{n}{r}$.

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4 Let x be a real number with $0<x<\pi$.Prove that, for all natural number n ,

$$sinx+\frac{sin3x}{3}+\frac{sin5x}{5}+\dots+\frac{sin(2n-1)x}{2n-1}>0.$$

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5 (a) The rectangle $PQRS$ with $PQ=l$ and $QR=m$ $(l,m\in\mathbb{N})$ is divided into lm unit squares. Prove that the diagonal PR intersects exactly $l+m-d$ of these squares, where $d=(l,m)$.
(b) A box with edge lengths $l,m,n\in\mathbb{N}$ is divided into lmn unit cubes. How many of the cubes does a main diagonal of the box intersect?
