Korea National Olympiad 2009

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Day 1

1 Let I, O be the incenter and the circumcenter of triangle ABC, and D, E, F be the circumcenters of triangle BIC, CIA, AIB. Let P, Q, R be the midpoints of segments DI, EI, FI. Prove that the circumcenter of triangle PQR, M, is the midpoint of segment IO.

2 Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{c(a^2+bc)} + \frac{b^3}{a(b^2+ca)} + \frac{c^3}{b(c^2+ab)} \geq \frac{3}{2}.$$

3 Let n be a positive integer. Suppose that the diophantine equation

$$z^n = 8x^{2009} + 23y^{2009}$$

uniquely has an integer solution $(x, y, z) = (0, 0, 0)$. Find the possible minimum value of n.

4 There are $n(\geq 3)$ students in a class. Some students are friends each other, and friendship is always mutual. There are $s(\geq 1)$ couples of two students who are friends, and $t(\geq 1)$ triples of three students who are each friends. For two students x, y define $d(x, y)$ be the number of students who are both friends with x and y. Prove that there exist three students u, v, w who are each friends and satisfying

$$d(u,v) + d(v,w) + d(w,u) \geq \frac{9t}{s}.$$

Day 2

1 Let $A = \{1, 2, 3, \cdots, 12\}$. Find the number of one-to-one function $f : A \to A$ satisfying following condition: for all $i \in A$, $f(i) - i$ is not a multiple of 3.

2 Let ABC be a triangle and $P, Q(\neq A, B, C)$ are the points lying on segments BC, CA. Let I, J, K be the incenters of triangle ABP, APQ, CPQ. Prove that $PIJK$ is a convex quadrilateral.

3 For all positive integer $n \geq 2$, prove that $2^n - 1$ can't be a divisor of $3^n - 1$.

4 For a positive integer n, define a function $f_n(x)$ at an interval $[0, n+1]$ as

$$f_n(x) = (\sum_{i=1}^{n} |x-i|)^2 - \sum_{i=1}^{n} (x-i)^2.$$

Let a_n be the minimum value of $f_n(x)$. Find the value of

$$\sum_{n=1}^{11} (-1)^{n+1} a_n.$$