Korea National Olympiad 2010

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by Ikeronalio

Day 1

1 Prove that $7^{2^{20}} + 7^{2^{19}} + 1$ has at least 21 distinct prime divisors.

2 Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\sqrt{a^2 + b^2 + \frac{1}{c^2}} + \sqrt{b^2 + c^2 + \frac{1}{a^2}} + \sqrt{c^2 + a^2 + \frac{1}{b^2}} \geq \sqrt{33}$$

3 Let I be the incenter of triangle ABC. The incircle touches BC, CA, AB at points P, Q, R. A circle passing through B, C is tangent to the circle I at point X, a circle passing through C, A is tangent to the circle I at point Y, and a circle passing through A, B is tangent to the circle I at point Z, respectively. Prove that three lines PX, QY, RZ are concurrent.

4 There are $n(\geq 4)$ people and some people shaked hands each other. Two people can shake hands at most 1 time. For arbitrary four people A, B, C, D such that $(A, B), (B, C), (C, D)$ shaked hands, then one of $(A, C), (A, D), (B, D)$ shaked hand each other. Prove the following statements.

(a) Prove that n people can be divided into two groups, $X, Y (\neq \emptyset)$, such that for all (x, y) where $x \in X$ and $y \in Y$, x and y shaked hands or x and y didn't shake hands.

(b) There exist two people A, B such that the set of people who are not A and B that shaked hands with A is same wiith the set of people who are not A and B that shaked hands with B.

Day 2

1 x, y, z are positive real numbers such that $x + y + z = 1$. Prove that

$$\sqrt{\frac{x}{1-x}} + \sqrt{\frac{y}{1-y}} + \sqrt{\frac{z}{1-z}} > 2$$

2 Let $ABCD$ be a cyclic convex quadrilateral. Let E be the intersection of lines AB, CD. P is the intersection of line passing B and perpendicular to AC, and line passing C and perpendicular to BD. Q is the intersection of line passing D and perpendicular to AC, and line passing A and perpendicular to BD. Prove that three points E, P, Q are collinear.

3 There are 2000 people, and some of them have called each other. Two people can call each other at most 1 time. For any two groups of three people A and B which $A \cap B = \emptyset$, there exist one person from each of A and B that haven't called each other. Prove that the number of two people called each other is less than 201000.

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4 There are 2010 people sitting around a round table. First, we give one person x a candy. Next, we give candies to 1 st person, $1+2$ th person, $1+2+3$ th person, $\cdots$, and $1+2+\cdots+2009$ th person clockwise from x. Find the number of people who get at least one candy.
