Korea National Olympiad 2011

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– Test 1

1 Two circles O, O' having same radius meet at two points, $A, B (A \neq B)$. Point P, Q are each on circle O and O' $(P \neq A, B\ Q \neq A, B)$. Select the point R such that $PAQR$ is a parallelogram. Assume that B, R, P, Q is cyclic. Now prove that $PQ = OO'$.

2 Let x, y be positive integers such that $\gcd(x, y) = 1$ and $x + 3y^2$ is a perfect square. Prove that $x^2 + 9y^4$ can't be a perfect square.

3 Let a, b, c, d real numbers such that $a + b + c + d = 19$ and $a^2 + b^2 + c^2 + d^2 = 91$. Find the maximum value of

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

4 Let k, n be positive integers. There are kn points $P_1, P_2, \cdots, P_{kn}$ on a circle. We can color each points with one of color $c_1, c_2, \cdots, c_k$. In how many ways we can color the points satisfying the following conditions?

(a) Each color is used n times.

(b) $\forall i \neq j$, if P_a and P_b is colored with color c_i , and P_c and P_d is colored with color c_j, then the segment P_aP_b and segment P_cP_d doesn't meet together.

– Test 2

1 Find the number of positive integer $n < 3^8$ satisfying the following condition.

"The number of positive integer $k (1 \leq k \leq \frac{n}{3})$ such that $\frac{n!}{(n-3k)! \cdot k! \cdot 3^{k+1}}$ is not a integer" is 216.

2 Let ABC be a triangle and its incircle meets BC, AC, AB at D, E and F respectively. Let point P on the incircle and inside $\triangle AEF$. Let $X = PB \cap DF, Y = PC \cap DE, Q = EX \cap FY$. Prove that the points A and Q lies on DP simultaneously or located opposite sides from DP.

3 There are n students each having r positive integers. Their nr positive integers are all different. Prove that we can divide the students into k classes satisfying the following conditions.

(a) $k \leq 4r$

(b) If a student A has the number m, then the student B in the same class can't have a number l such that

$$(m-1)! < l < (m+1)! + 1$$

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4 Let $x_1, x_2, \cdots, x_{25}$ real numbers such that $0 \le x_i \le i (i = 1, 2, \cdots, 25)$. Find the maximum value of

$$x_1^3 + x_2^3 + \cdots + x_{25}^3 - (x_1x_2x_3 + x_2x_3x_4 + \cdots x_{25}x_1x_2)$$
