Korea National Olympiad 2012

www.artofproblemsolving.com/community/c5344

by cadiTM, syk0526

Day 1

---

1 Let ABC be an obtuse triangle with $\angle A > 90^{\circ}$. Let circle O be the circumcircle of ABC. D is a point lying on segment AB such that $AD = AC$. Let AK be the diameter of circle O. Two lines AK and CD meet at L. A circle passing through D, K, L meets with circle O at $P(\neq K)$. Given that $AK = 2, \angle BCD = \angle BAP = 10^{\circ}$, prove that $DP = \sin(\frac{\angle A}{2})$.

---

2 There are n students $A_1, A_2, \cdots, A_n$ and some of them shaked hands with each other. (A_i and A_j can shake hands more than one time.) Let the student A_i shaked hands d_i times. Suppose $d_1 + d_2 + \cdots + d_n > 0$. Prove that there exist $1 \le i < j \le n$ satisfying the following conditions:
(a) Two students A_i and A_j shaked hands each other.
(b) $\frac{(d_1+d_2+\cdots+d_n)^2}{n^2} \le d_i d_j$

---

3 Find all triples (m, p, q) where m is a positive integer and p, q are primes.

$$2^m p^2 + 1 = q^5$$

---

4 a, b, c are positive numbers such that $a^2 + b^2 + c^2 = 2abc + 1$. Find the maximum value of

$$(a - 2bc)(b - 2ca)(c - 2ab)$$

Day 2

---

1 $p > 3$ is a prime number such that $p | 2^{p-1} - 1$ and $p \not| 2^x - 1$ for $x = 1, 2, \cdots, p-2$. Let $p = 2k + 3$. Now we define sequence $\{a_n\}$ as

$$a_i = a_{i+k} = 2^i (1 \le i \le k), \ a_{j+2k} = a_j a_{j+k} \ (j \ge 1)$$

Prove that there exist $2k$ consecutive terms of sequence $a_{x+1}, a_{x+2}, \cdots, a_{x+2k}$ such that for all $1 \le i < j \le 2k$, $a_{x+i} \not\equiv a_{x+j} \ (mod\ p)$.

---

2 Let w be the incircle of triangle ABC. Segments BC, CA meet with w at points D, E. A line passing through B and parallel to DE meets w at F and G. (F is nearer to B than G.) Line CG meets w at $H(\neq G)$. A line passing through G and parallel to EH meets with line AC at I.

Line IF meets with circle w at $J(\neq F)$. Lines CJ and EG meets at K. Let l be the line passing through K and parallel to JD. Prove that l, IF, ED meet at one point.

---

3 Let $\{a_1, a_2, \cdots, a_{10}\} = \{1, 2, \cdots, 10\}$. Find the maximum value of

$$\sum_{n=1}^{10}(na_n^2 - n^2a_n)$$

---

4 Let $p \equiv 3 \pmod 4$ be a prime. Define $T = \{(i,j) \mid i,j \in \{0,1,\cdots,p-1\}\} \setminus \{(0,0)\}$. For arbitrary subset $S(\neq \emptyset) \subset T$, prove that there exist subset $A \subset S$ satisfying following conditions:
(a) $(x_i, y_i) \in A (1 \leq i \leq 3)$ then $p \not| x_1 + x_2 - y_3$ or $p \not| y_1 + y_2 + x_3$.
(b) $8n(A) > n(S)$
