Korea National Olympiad 2013

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Day 1 November 10th

1 Let P be a point on segment BC. Q, R are points on AC, AB such that $PQ \parallel AB$ and $PR \parallel AC$. O, O_1, O_2 are the circumcenters of triangle ABC, BPR, PCQ. The circumcircles of BPR, PCQ meet at point $K(\neq P)$. Prove that $OO_1 = KO_2$.

2 Let $a, b, c > 0$ such that $ab + bc + ca = 3$. Prove that

$$\sum_{cyc} \frac{(a+b)^3}{(2(a+b)(a^2+b^2))^{\frac{1}{3}}} \geq 12$$

3 Prove that there exist monic polynomial $f(x)$ with degree of 6 and having integer coefficients such that
(1) For all integer m, $f(m) \neq 0$.
(2) For all positive odd integer n, there exist positive integer k such that $f(k)$ is divided by n.

4 $\{a_n\}$ is a positive integer sequence such that $a_{i+2} = a_{i+1} + a_i (i \geq 1)$. For positive integer n, define $\{b_n\}$ as

$$b_n = \frac{1}{a_{2n+1}} \sum_{i=1}^{4n-2} a_i$$

Prove that b_n is positive integer, and find the general form of b_n.

Day 2 November 10th

5 Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(mn) = \operatorname{lcm}(m, n) \cdot \gcd(f(m), f(n))$$

for all positive integer m, n.

6 Let O be circumcenter of triangle ABC. For a point P on segmet BC, the circle passing through P, B and tangent to line AB and the circle passing through P, C and tangent to line AC meet at point $Q(\neq P)$. Let D, E be foot of perpendicular from Q to AB, AC. ($D \neq B, E \neq C$) Two lines DE and BC meet at point R. Prove that O, P, Q are collinear if and only if A, R, Q are collinear.

7 For positive integer k, define integer sequence $\{b_n\},\{c_n\}$ as follows:

$$b_1=c_1=1$$

$$b_{2n}=kb_{2n-1}+(k-1)c_{2n-1}, c_{2n}=b_{2n-1}+c_{2n-1}$$
$$b_{2n+1}=b_{2n}+(k-1)c_{2n}, c_{2n+1}=b_{2n}+kc_{2n}$$

Let $a_k=b_{2014}$. Find the value of

$$\sum_{k=1}^{100}\left(a_k-\sqrt{{a_k}^2-1}\right)^{\frac{1}{2014}}$$

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8 For positive integer a,b,c,d there are $a+b+c+d$ points on plane which none of three are collinear. Prove there exist two lines l_1,l_2 such that
(1) l_1,l_2 are not parallel.
(2) l_1,l_2 do not pass through any of $a+b+c+d$ points.
(3) There are a,b,c,d points on each region separated by two lines l_1,l_2.
