**Tuymaada Olympiad 2017**

www.artofproblemsolving.com/community/c534575

by Snakes, cloudRo, AngleChasingXD, parmenides51

– Juniors

– **Day 1**

**1** Functions $f$ and $g$ are defined on the set of all integers in the interval $[-100; 100]$ and take integral values. Prove that for some integral $k$ the number of solutions of the equation $f(x) - g(y) = k$ is odd.
( A. Golovanov)

**2** $ABCD$ is a cyclic quadrilateral such that the diagonals $AC$ and $BD$ are perpendicular and their intersection is $P$. Point $Q$ on the segment $CP$ is such that $CQ = AP$. Prove that the perimeter of triangle $BDQ$ is at least $2AC$.

Tuymaada 2017 Q2 Juniors

**3** In a country every 2 cities are connected either by a direct bus route or a direct plane flight. A $clique$ is a set of cities such that every 2 cities in the set are connected by a direct flight. A $cluque$ is a set of cities such that every 2 cities in the set are connected by a direct flight, and every 2 cities in the set are connected to the same number of cities by a bus route. A $claque$ is a set of cities such that every 2 cities in the set are connected by a direct flight, and every 2 numbers of bus routes from a city in the set are different. Prove that the number of cities of any clique is at most the product of the biggest possible number of cities in a cluque and the the biggest possible number of cities in a claque.

Tuymaada 2017 Q3 Juniors

**4** A right triangle has all its sides rational numbers and the area $S$. Prove that there exists a right triangle, different from the original one, such that all its sides are rational numbers and its area is $S$.

Tuymaada 2017 Q4 Juniors

– **Day 2**

**5** $BL$ is the bisector of an isosceles triangle $ABC$. A point $D$ is chosen on the Base $BC$ and a point $E$ is chosen on the lateral side $AB$ so that $AE = \frac{1}{2}AL = CD$. Prove that $LE = LD$.

Tuymaada 2017 Q5 Juniors

**6** Let $\sigma(n)$ denote the sum of positive divisors of a number $n$. A positive integer $N = 2^r b$ is given,where $r$ and $b$ are positive integers and $b$ is odd. It is known that $\sigma(N) = 2N - 1$. Prove that $b$ and $\sigma(b)$ are coprime.

Tuymaada Q6 Juniors

---

**7** An equilateral triangle with side $20$ is divided by there series of parallel lines into $400$ equilateral triangles with side $1$. What maximum number of these small triangles can be crossed (internally) by one line?

Tuymaada 2017 Q7 Juniors

---

**8** We consider the graph with vertices $A_1, A_2, \dots A_{2015}$ , $B_1, B_2, \dots B_{2015}$ and edges $A_iA_{i+1}, A_iB_i, B_iB_{i+17}$, taken cyclicaly. Is it true that 4 cops can catch a robber on this graph for every initial position?( First the 4 cops make a move, then the robber makes a move, then the cops make a move etc. A move consists of jumping from the vertex you stay on an adiacent vertex or by staying on your current vertex. Everyone knows the position of everyone everytime. The cops can coordinate their moves. The robber is caught when he shares the same vertex with a cop.)

Tuymaada 2017 Q8 Juniors

---

**–** Seniors

---

**–** **Day 1**

---

**1** Same as junior Q1

---

**2** Same as junior Q2

---

**3** Same as junior Q4

---

**4** There are 25 masks of different colours. k sages play the following game. They are shown all the masks. Then the sages agree on their strategy. After that the masks are put on them so that each sage sees the masks on the others but can not see who wears each mask and does not see his own mask. No communication is allowed. Then each of them simultaneously names one colour trying to guess the colour of his mask. Find the minimum k for which the sages can agree so that at least one of them surely guesses the colour of his mask.
( S. Berlov )

---

**–** **Day 2**

---

**5** Does there exist a quadratic trinomial $f(x)$ such that $f(1/2017) = 1/2018$, $f(1/2018) = 1/2017$, and two of its coefficients are integers?

(A. Khrabrov)

 

**6** Let $\sigma(n)$ denote the sum of positive divisors of a number $n$. A positive integer $N = 2^r b$ is given, where $r$ and $b$ are positive integers and $b$ is odd. It is known that $\sigma(N) = 2N - 1$. Prove that $b$ and $\sigma(b)$ are coprime.

(J. Antalan, J. Dris)

**7** A point $E$ lies on the extension of the side $AD$ of the rectangle $ABCD$ over $D$. The ray $EC$ meets the circumcircle $\omega$ of $ABE$ at the point $F \neq E$. The rays $DC$ and $AF$ meet at $P$. $H$ is the foot of the perpendicular drawn from $C$ to the line $\ell$ going through $E$ and parallel to $AF$. Prove that the line $PH$ is tangent to $\omega$.

(A. Kuznetsov)

**8** Two points $A$ and $B$ are given in the plane. A point $X$ is called their *preposterous midpoint* if there is a Cartesian coordinate system in the plane such that the coordinates of $A$ and $B$ in this system are non-negative, the abscissa of $X$ is the geometric mean of the abscissae of $A$ and $B$, and the ordinate of $X$ is the geometric mean of the ordinates of $A$ and $B$. Find the locus of all the *preposterous midpoints* of $A$ and $B$.

(K. Tyschu)