Czech-Polish-Slovak Match 2015

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by Snakes, Radar

–	**Day 1**
1	On a circle of radius r, the distinct points A, B, C, D, and E lie in this order, satisfying $AB = CD = DE > r$. Show that the triangle with vertices lying in the centroids of the triangles ABD, BCD, and ADE is obtuse. *Proposed by Tom Jurk, Slovakia*
2	A family of sets F is called perfect if the following condition holds: For every triple of sets $X_1, X_2, X_3 \in F$, at least one of the sets $$(X_1 \setminus X_2) \cap X_3,$$ $$(X_2 \setminus X_1) \cap X_3$$ is empty. Show that if F is a perfect family consisting of some subsets of a given finite set U, then $\lvert F\rvert \le \lvert U\rvert + 1$. *Proposed by Micha Pilipczuk*
3	Real numbers x, y, z satisfy $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + x + y + z = 0$$ and none of them lies in the open interval $(-1, 1)$. Find the maximum value of $x + y + z$. *Proposed by Jaromr ima*
–	**Day 2**
1	A strange calculator has only two buttons with positive itegers, each of them consisting of two digits. It displays the number 1 at the beginning. Whenever a button with number N is pressed, the calculator replaces the displayed number X with the number $X \cdot N$ or $X + N$. Multiplication and addition alternate, multiplication is the first. (For example,if the number 10 is on the 1st button, the number 20 is on the 2nd button, and we consecutively press the 1st, 2nd, 1st and 1st button, we get the results $1 \cdot 10 = 10$, $10 + 20 = 30$, $30 \cdot 10 = 300$, and $300 + 10 = 310$.) Decide whether there exist particular values of the two-digit nubers on the buttons such that one can display infinitely many numbers (without cleaning the display, i.e. you must keep going and get infinitel many numbers) ending with (a) 2015, (b) 5813.

Proposed by Michal Rolnek and Peter Novotn

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2 Let ABC be an acute triangle, which is not equilateral. Denote by O and H its circumcenter and orthocenter, respectively. The circle k passes through B and touches the line AC at A. The circle l with center on the ray BH touhes the line AB at A. The circles k and l meet in X ($X \neq A$). Show that $\angle HXO = 180^\circ - \angle BAC$.

Proposed by Josef Tkadlec

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3 Let n be even positive integer. There are n real positive numbers written on the blackboard. In every step, we choose two numbers, erase them, and replace *each* of then by their product. Show that for any initial n-tuple it is possible to obtain n equal numbers on the blackboard after a finite number of steps.

Proposed by Peter Novotn
