Czech-Polish-Slovak Match 2016

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by Snakes, MRF2017, parmenides51

Day 1

1 Let P be a non-degenerate polygon with n sides, where $n > 4$. Prove that there exist three distinct vertices A, B, C of P with the following property:If ℓ_1, ℓ_2, ℓ_3 are the lengths of the three polygonal chains into which A, B, C break the perimeter of P, then there is a triangle with side lengths ℓ_1, ℓ_2 and ℓ_3.
Remark: By a non-degenerate polygon we mean a polygon in which every two sides are disjoint, apart from consecutive ones, which share only the common endpoint.(Poland)

2 Let $m, n > 2$ be even integers. Consider a board of size $m \times n$ whose every cell is colored either black or white. The Guesser does not see the coloring of the board but may ask the Oracle some questions about it. In particular, she may inquire about two adjacent cells (sharing an edge) and the Oracle discloses whether the two adjacent cells have the same color or not. The Guesser eventually wants to find the parity of the number of adjacent cell-pairs whose colors are different. What is the minimum number of inquiries the Guesser needs to make so that she is guaranteed to find her answer?

(Czech Republic)

3 Let n be a positive integer. For a fi nite set M of positive integers and each $i \in \{0, 1, ..., n-1\}$, we denote s_i the number of non-empty subsets of M whose sum of elements gives remainder i after division by n. We say that M is "n-balanced" if $s_0 = s_1 = ... = s_{n-1}$. Prove that for every odd number n there exists a non-empty n-balanced subset of $\{0, 1, ..., n\}$.
For example if $n = 5$ and $M = \{1, 3, 4\}$, we have $s_0 = s_1 = s_2 = 1, s_3 = s_4 = 2$ so M is not 5-balanced.(Czech Republic)

Day 2

1 Find all quadruplets (a, b, c, d) of real numbers satisfying the system
$(a+b)(a^2+b^2) = (c+d)(c^2+d^2)$
$(a+c)(a^2+c^2) = (b+d)(b^2+d^2)$
$(a+d)(a^2+d^2) = (b+c)(b^2+c^2)$
(Slovakia)

2 Prove that for every non-negative integer n there exist integers x, y, z with $gcd(x, y, z) = 1$, such that $x^2 + y^2 + z^2 = 3^{2^n}$.(Poland)

3 Let ABC be an acute-angled triangle with $AB < AC$. Tangent to its circumcircle Ω at A intersects the line BC at D. Let G be the centroid of $\triangle ABC$ and let AG meet Ω again at $H \neq A$. Suppose the line DG intersects the lines AB and AC at E and F, respectively. Prove that $\angle EHG = \angle GHF$.(Slovakia)