Korea National Olympiad 2014

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Day 1

1 For x, y positive integers, $x^2 - 4y + 1$ is a multiple of $(x - 2y)(1 - 2y)$. Prove that $|x - 2y|$ is a square number.

2 Determine all the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfies the following.

$$f(xf(x) + f(x)f(y) + y - 1) = f(xf(x) + xy) + y - 1$$

3 AB is a chord of O and AB is not a diameter of O. The tangent lines to O at A and B meet at C. Let M and N be the midpoint of the segments AC and BC, respectively. A circle passing through C and tangent to O meets line MN at P and Q. Prove that $\angle PCQ = \angle CAB$.

4 There is a city with n metro stations, each located at a vertex of a regular n-polygon. Metro Line 1 is a line which only connects two non-neighboring stations A and B. Metro Line 2 is a cyclic line which passes through all the stations in a shape of regular n-polygon. For each line metro can run in any direction, and A and B are the stations which one can transfer into other line. The line between two neighboring stations is called 'metro interval'. For each station there is one stationmaster, and there are at least one female stationmaster and one male stationmaster. If n is odd, prove that for any integer k $(0 < k < n)$ there is a path that starts from a station with a male stationmaster and ends at a station with a female stationmaster, passing through k metro intervals.

Day 2

1 There is a convex quadrilateral $ABCD$ which satisfies $\angle A = \angle D$.
Let the midpoints of AB, AD, CD be L, M, N.
Let's say the intersection point of AC, BD be E .
Let's say point F which lies on $\overrightarrow{ME}$ satisfies $\overline{ME} \times \overline{MF} = \overline{MA}^2$.

Prove that $\angle LFM = \angle MFN$. :)

2 How many one-to-one functions $f : \{1, 2, \cdots, 9\} \to \{1, 2, \cdots, 9\}$ satisfy (i) and (ii)?
(i) $f(1) > f(2)$, $f(9) < 9$.
(ii) For each $i = 3, 4, \cdots, 8$, if $f(1), \cdots, f(i-1)$ are smaller than $f(i)$, then $f(i+1)$ is also smaller than $f(i)$.

3 Let x, y, z be the real numbers that satisfies the following.

$(x-y)^2+(y-z)^2+(z-x)^2=8, x^3+y^3+z^3=1$

Find the minimum value of $x^4+y^4+z^4$.

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4 Prove that there exists a function $f:\mathbb{N}\rightarrow\mathbb{N}$ that satisfies the following

(1) $\{f(n):n\in\mathbb{N}\}$ is a finite set; and
(2) For nonzero integers $x_1,x_2,\ldots,x_{1000}$ that satisfy $f(|x_1|)=f(|x_2|)=\cdots=f(|x_{1000}|)$, then $x_1+2x_2+2^2x_3+2^3x_4+2^4x_5+\cdots+2^{999}x_{1000}\neq 0$.
