France Team Selection Test 2000

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Day 1

1 Points P, Q, R, S lie on a circle and $\angle PSR$ is right. H, K are the projections of Q on lines PR, PS. Prove that HK bisects segment QS.

2 A function from the positive integers to the positive integers satisfies these properties
1. $f(ab) = f(a)f(b)$ for any two coprime positive integers a, b.
2. $f(p+q) = f(p) + f(q)$ for any two primes p, q.
Prove that $f(2) = 2, f(3) = 3, f(1999) = 1999$.

3 a, b, c, d are positive reals with sum 1. Show that $\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2}$ with equality iff $a = b = c = d = \frac{1}{4}$.

Day 2

1 Some squares of a 1999×1999 board are occupied with pawns. Find the smallest number of pawns for which it is possible that for each empty square, the total number of pawns in the row or column of that square is at least 1999.

2 A, B, C, D are points on a circle in that order. Prove that $|AB - CD| + |AD - BC| \geq 2|AC - BD|$.

3 Find all nonnegative integers x, y, z such that $(x+1)^{y+1} + 1 = (x+2)^{z+1}$.