CentroAmerican 2017

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Day 1

1 The figure below shows a hexagonal net formed by many congruent equilateral triangles. Taking turns, Gabriel and Arnaldo play a game as follows. On his turn, the player colors in a segment, including the endpoints, following these three rules:

i) The endpoints must coincide with vertices of the marked equilateral triangles.

ii) The segment must be made up of one or more of the sides of the triangles.

iii) The segment cannot contain any point (endpoints included) of a previously colored segment.

Gabriel plays first, and the player that cannot make a legal move loses. Find a winning strategy and describe it.

2 We call a pair (a, b) of positive integers, $a < 391$, *pupusa* if

$$\text{lcm}(a, b) > \text{lcm}(a, 391)$$

Find the minimum value of b across all *pupusa* pairs.

Fun Fact: OMCC 2017 was held in El Salvador. *Pupusa* is their national dish. It is a corn tortilla filled with cheese, meat, etc.

3 Let ABC be a triangle and D be the foot of the altitude from A. Let l be the line that passes through the midpoints of BC and AC. E is the reflection of D over l. Prove that the circumcentre of $\triangle ABC$ lies on the line AE.

Day 2

1 ABC is a right-angled triangle, with $\angle ABC = 90°$. B' is the reflection of B over AC. M is the midpoint of AC. We choose D on $\overrightarrow{BM}$, such that $BD = AC$. Prove that $B'C$ is the angle bisector of $\angle MB'D$.

NOTE: An important condition not mentioned in the original problem is $AB < BC$. Otherwise, $\angle MB'D$ is not defined or $B'C$ is the external bisector.

2 Susana and Brenda play a game writing polynomials on the board. Susana starts and they play taking turns.

1) On the preparatory turn (turn 0), Susana choose a positive integer n_0 and writes the polynomial $P_0(x) = n_0$.

2) On turn 1, Brenda choose a positive integer n_1, different from n_0, and either writes the polynomial

$$P_1(x) = n_1x + P_0(x) \text{ or } P_1(x) = n_1x - P_0(x)$$

3) In general, on turn k, the respective player chooses an integer n_k, different from $n_0, n_1, \ldots, n_{k-1}$, and either writes the polynomial

$$P_k(x) = n_kx^k + P_{k-1}(x) \text{ or } P_k(x) = n_kx^k - P_{k-1}(x)$$

The first player to write a polynomial with at least one whole whole number root wins. Find and describe a winning strategy.

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3 Tita the Frog sits on the number line. She is initially on the integer number $k > 1$. If she is sitting on the number n, she hops to the number $f(n) + g(n)$, where $f(n)$ and $g(n)$ are, respectively, the biggest and smallest positive prime numbers that divide n. Find all values of k such that Tita can hop to infinitely many distinct integers.
