Czech-Polish-Slovak Match 2013

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Day 1

1 Suppose $ABCD$ is a cyclic quadrilateral with $BC = CD$. Let ω be the circle with center C tangential to the side BD. Let I be the centre of the incircle of triangle ABD. Prove that the straight line passing through I, which is parallel to AB, touches the circle ω.

2 Prove that for every real number $x > 0$ and each integer $n > 0$ we have

$$x^n + \frac{1}{x^n} - 2 \geq n^2 \left(x + \frac{1}{x} - 2 \right)$$

3 For each rational number r consider the statement: If x is a real number such that $x^2 - rx$ and $x^3 - rx$ are both rational, then x is also rational.

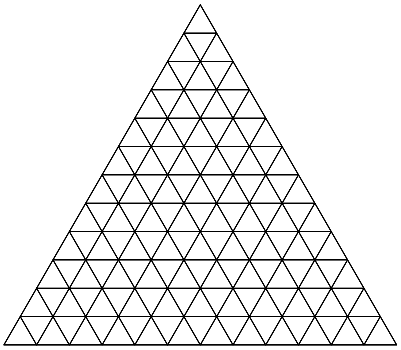
(a) Prove the claim for $r \geq \frac{4}{3}$ and $r \leq 0$.
(b) Let p, q be different odd primes such that $3p < 4q$. Prove that the claim for $r = \frac{p}{q}$ does not hold.

Day 2

1 Let a and b be integers, where b is not a perfect square. Prove that $x^2 + ax + b$ may be the square of an integer only for finite number of integer values of x.

(Martin Panák)

2 Triangular grid divides an equilateral triangle with sides of length n into n^2 triangular cells as shown in figure for $n = 12$. Some cells are infected. A cell that is not yet infected, ia infected when it shares adjacent sides with at least two already infected cells. Specify for $n = 12$, the least number of infected cells at the start in which it is possible that over time they will infected all the cells of the original triangle.



3 Let ABC be a triangle inscribed in a circle. Point P is the center of the arc BAC. The circle with the diameter CP intersects the angle bisector of angle $\angle BAC$ at points K, L $(|AK| < |AL|)$. Point M is the reflection of L with respect to line BC. Prove that the circumcircle of the triangle BKM passes through the center of the segment BC .