France Team Selection Test 2002

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Day 1

1 In an acute-angled triangle ABC, A_1 and B_1 are the feet of the altitudes from A and B respectively, and M is the midpoint of AB.
a) Prove that MA_1 is tangent to the circumcircle of triangle A_1B_1C.
b) Prove that the circumcircles of triangles A_1B_1C, BMA_1, and AMB_1 have a common point.

2 Consider the set S of integers k which are products of four distinct primes. Such an integer $k = p_1p_2p_3p_4$ has 16 positive divisors $1 = d_1 < d_2 < \ldots < d_{15} < d_{16} = k$. Find all elements of S less than 2002 such that $d_9 - d_8 = 22$.

3 Let n be a positive integer and let $(a_1, a_2, \ldots, a_{2n})$ be a permutation of $1, 2, \ldots, 2n$ such that the numbers $|a_{i+1} - a_i|$ are pairwise distinct for $i = 1, \ldots, 2n - 1$.
Prove that $\{a_2, a_4, \ldots, a_{2n}\} = \{1, 2, \ldots, n\}$ if and only if $a_1 - a_{2n} = n$.

Day 2

1 There are three colleges in a town. Each college has n students. Any student of any college knows $n + 1$ students of the other two colleges. Prove that it is possible to choose a student from each of the three colleges so that all three students would know each other.

2 Let ABC be a non-equilateral triangle. Denote by I the incenter and by O the circumcenter of the triangle ABC. Prove that $\angle AIO \leq \frac{\pi}{2}$ holds if and only if $2 \cdot BC \leq AB + AC$.

3 Let $p \geq 3$ be a prime number. Show that there exist p positive integers $a_1, a_2, \ldots, a_p$ not exceeding $2p^2$ such that the $\frac{p(p-1)}{2}$ sums $a_i + a_j$ $(i < j)$ are all distinct.