France Team Selection Test 2003

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Day 1

1 A lattice point in the coordinate plane with origin O is called invisible if the segment OA contains a lattice point other than O, A. Let L be a positive integer. Show that there exists a square with side length L and sides parallel to the coordinate axes, such that all points in the square are invisible.

2 A lattice point in the coordinate plane with origin O is called invisible if the segment OA contains a lattice point other than O, A. Let L be a positive integer. Show that there exists a square with side length L and sides parallel to the coordinate axes, such that all points in the square are invisible.

3 M is an arbitrary point inside $\triangle ABC$. AM intersects the circumcircle of the triangle again at A_1. Find the points M that minimise $\frac{MB \cdot MC}{MA_1}$.

Day 2

1 Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.

2 10 cities are connected by one-way air routes in a way so that each city can be reached from any other by several connected flights. Let n be the smallest number of flights needed for a tourist to visit every city and return to the starting city. Clearly n depends on the flight schedule. Find the largest n and the corresponding flight schedule.

3 Let $p_1, p_2, \dots, p_n$ be distinct primes greater than 3. Show that $2^{p_1 p_2 \cdots p_n} + 1$ has at least 4^n divisors.