France Team Selection Test 2004

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Day 1

1 If n is a positive integer, let $A=\{n,n+1,...,n+17\}$.
Does there exist some values of n for which we can divide A into two disjoints subsets B and C such that the product of the elements of B is equal to the product of the elements of C?

2 Let $ABCD$ be a parallelogram. Let M be a point on the side AB and N be a point on the side BC such that the segments AM and CN have equal lengths and are non-zero. The lines AN and CM meet at Q.
Prove that the line DQ is the bisector of the angle $\measuredangle ADC$.

Alternative formulation. Let $ABCD$ be a parallelogram. Let M and N be points on the sides AB and BC, respectively, such that $AM=CN\neq 0$. The lines AN and CM intersect at a point Q.
Prove that the point Q lies on the bisector of the angle $\measuredangle ADC$.

3 Each point of the plane with two integer coordinates is the center of a disk with radius $\frac{1}{1000}$.
Prove that there exists an equilateral triangle whose vertices belong to distinct disks.
Prove that such a triangle has side-length greater than 96.

Day 2

1 Let n be a positive integer, and $a_1,...,a_n,b_1,...,b_n$ be $2n$ positive real numbers such that $a_1+...+a_n=b_1+...+b_n=1$.

Find the minimal value of $\frac{a_1^2}{a_1+b_1}+\frac{a_2^2}{a_2+b_2}+...+\frac{a_n^2}{a_n+b_n}$.

2 Let P, Q, and R be the points where the incircle of a triangle ABC touches the sides AB, BC, and CA, respectively.
Prove the inequality $\frac{BC}{PQ}+\frac{CA}{QR}+\frac{AB}{RP}\geq 6$.

3 Let P be the set of prime numbers. Consider a subset M of P with at least three elements. We assume that, for each non empty and finite subset A of M, with $A\neq M$, the prime divisors of the integer $(\prod_{p\in A})-1$ belong to M.
Prove that $M=P$.