France Team Selection Test 2005

www.artofproblemsolving.com/community/c5351

by Igor, El Chamo, darij grinberg, remo, cyberjoe, Valentin Vornicu

Day 1 May 26th

1 Let x, y be two positive integers such that $3x^2+x=4y^2+y$.
Prove that $x-y$ is a perfect square.

2 Two right angled triangles are given, such that the incircle of the first one is equal to the circumcircle of the second one. Let S (respectively S') be the area of the first triangle (respectively of the second triangle).

Prove that $\frac{S}{S'}\geq 3+2\sqrt{2}$.

3 In an international meeting of $n \geq 3$ participants, 14 languages are spoken. We know that:

- Any 3 participants speak a common language.

- No language is spoken more that by the half of the participants.

What is the least value of n?

Day 2 May 27th

4 Let X be a non empty subset of $\mathbb{N} = \{1,2,\ldots\}$. Suppose that for all $x \in X$, $4x \in X$ and $\lfloor \sqrt{x} \rfloor \in X$. Prove that $X=\mathbb{N}$.

5 Let ABC be a triangle such that $BC=AC+\frac{1}{2}AB$. Let P be a point of AB such that $AP=3PB$.

Show that $\widehat{PAC} = 2 \widehat{CPA}$.

6 Let P be a polynom of degree $n \geq 5$ with integer coefficients given by $P(x)=a_{n}x^n+a_{n-1}x^{n-1}+\cdots+a_0 \quad$ with $a_i \in \mathbb{Z}$, $a_n \neq 0$.

Suppose that P has n different integer roots (elements of $\mathbb{Z}$) : $0,\alpha_2,\ldots,\alpha_n$. Find all integers $k \in \mathbb{Z}$ such that $P(P(k))=0$.