France Team Selection Test 2007

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Day 1

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1 For a positive integer a, a' is the integer obtained by the following method: the decimal writing of a' is the inverse of the decimal writing of a (the decimal writing of a' can begin by zeros, but not the one of a); for instance if $a = 2370$, $a' = 0732$, that is 732.

Let a_1 be a positive integer, and $(a_n)_{n\geq 1}$ the sequence defined by a_1 and the following formula for $n \geq 1$:

$$a_{n+1} = a_n + a'_n.$$

Can a_7 be prime?

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2 Let a, b, c, d be positive reals such taht $a + b + c + d = 1$.

Prove that:

$$6(a^3 + b^3 + c^3 + d^3) \geq a^2 + b^2 + c^2 + d^2 + \frac{1}{8}.$$

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3 Let A, B, C, D be four distinct points on a circle such that the lines (AC) and (BD) intersect at E, the lines (AD) and (BC) intersect at F and such that (AB) and (CD) are not parallel.

Prove that C, D, E, F are on the same circle if, and only if, $(EF)\perp(AB)$.

Day 2

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1 Do there exist 5 points in the space, such that for all $n \in \{1, 2, \dots, 10\}$ there exist two of them at distance between them n?

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2 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$:

$$f(x - y + f(y)) = f(x) + f(y).$$

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3 A point D is chosen on the side AC of a triangle ABC with $\angle C < \angle A < 90^\circ$ in such a way that $BD = BA$. The incircle of ABC is tangent to AB and AC at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that the line KL intersects the line segment AJ at its midpoint.
