France Team Selection Test 2012

www.artofproblemsolving.com/community/c5354

by WakeUp

Day 1 March 10th

1 Let n and k be two positive integers. Consider a group of k people such that, for each group of n people, there is a $(n+1)$-th person that knows them all (if A knows B then B knows A).
1) If $k=2n+1$, prove that there exists a person who knows all others.
2) If $k=2n+2$, give an example of such a group in which no-one knows all others.

2 Let ABC be an acute-angled triangle with $AB\neq AC$. Let Γ be the circumcircle, H the orthocentre and O the centre of Γ. M is the midpoint of BC. The line AM meets Γ again at N and the circle with diameter AM crosses Γ again at P. Prove that the lines AP, BC, OH are concurrent if and only if $AH=HN$.

3 Let p be a prime number. Find all positive integers $a,b,c\ge 1$ such that:

$$a^p+b^p=p^c.$$

Day 2 March 11th

1 Let $k>1$ be an integer. A function $f:\mathbb{N}^*\to\mathbb{N}^*$ is called k-*tastrophic* when for every integer $n>0$, we have $f_k(n)=n^k$ where f_k is the k-th iteration of f:

$$f_k(n)=\underbrace{f\circ f\circ\cdots\circ f}_{k\text{ times}}(n)$$

For which k does there exist a k-tastrophic function?

2 Determine all non-constant polynomials $X^n+a_{n-1}X^{n-1}+\cdots+a_1X+a_0$ with integer coefficients for which the roots are exactly the numbers $a_0,a_1,\ldots,a_{n-1}$ (with multiplicity).

3 Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA and AB. Prove that the midpoint of the segment EF lies on the line through the two intersections of ω_E and ω_F.

Proposed by Carlos Yuzo Shine, Brazil