France Team Selection Test 2014

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1 Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \le a_i \le 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

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2 Two circles O_1 and O_2 intersect each other at M and N. The common tangent to two circles nearer to M touch O_1 and O_2 at A and B respectively. Let C and D be the reflection of A and B respectively with respect to M. The circumcircle of the triangle DCM intersect circles O_1 and O_2 respectively at points E and F (both distinct from M). Show that the circumcircles of triangles MEF and NEF have same radius length.

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3 Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

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4 Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

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5 Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

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6 Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be positive reals. Show that there are numbers $a_1, a_2, \ldots, a_n \in \{-1, 1\}$ such that the following holds:

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2 \ge (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2$$
