National Olympiad 11-12 2006

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by Sasha

Day 1

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1 Let $n\in\mathbb{N}^*$. Prove that

$$\lim_{x\to 0}\frac{(1+x^2)^{n+1}-\prod_{k=1}^{n}\cos kx}{x\sum_{k=1}^{n}\sin kx}=\frac{2n^2+n+12}{6n}.$$

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2 Function $f:[a,b]\to\mathbb{R}$, $0<a<b$ is continuous on $[a,b]$ and differentiable on (a,b). Prove that there exists $c\in(a,b)$ such that

$$f'(c)=\frac{1}{a-c}+\frac{1}{b-c}+\frac{1}{a+b}.$$

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3 On each of the 2006 cards a natural number is written. Cards are placed arbitrarily in a row. 2 players take in turns a card from any end of the row until all the cards are taken. After that each player calculates sum of the numbers written of his cards. If the sum of the first player is not less then the sum of the second one then the first player wins. Show that there's a winning strategy for the first player.

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4 Let $ABCDE$ be a right quadrangular pyramid with vertex E and height EO. Point S divides this height in the ratio $ES:SO=m$. In which ratio does the plane (ABC) divide the lateral area of the pyramid.

Day 2

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5 Let $n\in\mathbb{N}^*$. Solve the equation $\sum_{k=0}^{n}C_n^k\cos 2kx=\cos nx$ in $\mathbb{R}$.

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6 Sequences $(x_n)_{n\ge1}$, $(y_n)_{n\ge1}$ satisfy the relations $x_n=4x_{n-1}+3y_{n-1}$ and $y_n=2x_{n-1}+3y_{n-1}$ for $n\ge1$. If $x_1=y_1=5$ find x_n and y_n.

Calculate $\lim_{n\to\infty}\frac{x_n}{y_n}$.

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7 Let $n \in \mathbb{N}^*$. $2n+3$ points on the plane are given so that no 3 lie on a line and no 4 lie on a circle. Is it possible to find 3 points so that the interior of the circle passing through them would contain exactly n of the remaining points.

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8 Given an alfabet of n letters. A sequence of letters such that between any 2 identical letters there are no 2 identical letters is called a *word*.

a) Find the maximal possible length of a *word*.

b) Find the number of the *words* of maximal length.
