National Olympiad 11-12 2008

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by freemind

Day 1	March 1st	
1	Consider the equation $x^4-4x^3+4x^2+ax+b=0$, where $a,b\in\mathbb{R}$. Determine the largest value $a+b$ can take, so that the given equation has two distinct positive roots x_1,x_2 so that $x_1+x_2=2x_1x_2$.	
2	Find the exact value of $E=\int_0^{\frac{\pi}{2}}\cos^{1003}x\,\mathrm{d}x\cdot\int_0^{\frac{\pi}{2}}\cos^{1004}x\,\mathrm{d}x\cdot$.	
3	In the usual coordinate system xOy a line d intersect the circles $C_1:(x+1)^2+y^2=1$ and $C_2:(x-2)^2+y^2=4$ in the points A,B,C and D (in this order). It is known that $A\left(-\frac{3}{2},\frac{\sqrt{3}}{2}\right)$ and $\angle BOC=60^\circ$. All the Oy coordinates of these 4 points are positive. Find the slope of d.	
4	Define the sequence $(a_p)_{p\ge0}$ as follows: $a_p=\frac{\binom{p}{0}}{2\cdot4}-\frac{\binom{p}{1}}{3\cdot5}+\frac{\binom{p}{2}}{4\cdot6}-\ldots+(-1)^p\cdot\frac{\binom{p}{p}}{(p+2)(p+4)}$. Find $\lim_{n\to\infty}(a_0+a_1+\ldots+a_n)$.	
Day 2	March 2nd	
5	Find the least positive integer n so that the polynomial $P(X)=\sqrt{3}\cdot X^{n+1}-X^n-1$ has at least one root of modulus 1.	
6	Find $\lim_{n\to\infty}a_n$ where $(a_n)_{n\ge1}$ is defined by $a_n=\frac{1}{\sqrt{n^2+8n-1}}+\frac{1}{\sqrt{n^2+16n-1}}+\frac{1}{\sqrt{n^2+24n-1}}+\ldots+\frac{1}{\sqrt{9n^2-1}}$.	
7	Triangle ABC has fixed vertices B and C, so that $BC=2$ and A is variable. Denote by H and G the orthocenter and the centroid, respectively, of triangle ABC. Let $F\in(HG)$ so that $\frac{HF}{FG}=3$. Find the locus of the point A so that $F\in BC$.	
8	Evaluate $I=\int_0^{\frac{\pi}{4}}\left(\sin^6 2x+\cos^6 2x\right)\cdot\ln(1+\tan x)\mathrm{d}x$.	