Taiwan National Olympiad 1994

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Day 1

1 Let $ABCD$ be a quadrilateral with $AD=BC$ and $\widehat{A}+\widehat{B}=120^{0}$. Let us draw equilateral ACP,DCQ,DBR away from AB . Prove that the points P,Q,R are collinear.

2 Let a,b,c are positive real numbers and α be any real number. Denote $f(\alpha)=abc(a^{\alpha}+b^{\alpha}+c^{\alpha}),g(\alpha)=a^{2+\alpha}(b+c-a)+b^{2+\alpha}(-b+c+a)+c^{2+\alpha}(b-c+a)$. Determine $\min|f(\alpha)-g(\alpha)|$ and $\max|f(\alpha)-g(\alpha)|$, if they are exists.

3 Let a be a positive integer such that $5^{1994}-1\mid a$. Prove that the expression of a in base 5 contains at least 1994 nonzero digits.

Day 2

4 Prove that there are infinitely many positive integers n with the following property: For any n integers $a_{1},a_{2},...,a_{n}$ which form in arithmetic progression, both the mean and the standard deviation of the set $\{a_{1},a_{2},...,a_{n}\}$ are integers.
Remark. The mean and standard deviation of the set $\{x_{1},x_{2},...,x_{n}\}$ are defined by $\overline{x}=\frac{x_{1}+x_{2}+...+x_{n}}{n}$ and $\sqrt{\frac{\sum (x_{i}-\overline{x})^{2}}{n}}$, respectively.

5 Given $X=\{0,a,b,c\}$, let $M(X)=\{f|f: X\to X\}$ denote the set of all functions from X into itself. An addition table on X is given us follows: $+ \ 0\ a\ b\ c\ 0\ 0\ a\ b\ c\ a\ a\ 0\ c\ b\ b\ b\ c\ 0\ a\ c\ c\ b\ a\ 0$
a)If $S=\{f\in M(X)|f(x+y+x)=f(x)+f(y)+f(x)\forall x,y\in X\}$, find $|S|$.
b)If $I=\{f\in M(X)|f(x+x)=f(x)+f(x)\forall x\in X\}$, find $|I|$.

6 For $-1\le x\le 1$ and $n\in\mathbb{N}$ define $T_{n}(x)=\frac{1}{2^{n}}[(x+\sqrt{1-x^{2}})^{n}+(x-\sqrt{1-x^{2}})^{n}]$.
a)Prove that T_{n} is a monic polynomial of degree n in x and that the maximum value of $|T_{n}(x)|$ is $\frac{1}{2^{n-1}}$.
b)Suppose that $p(x)=x^{n}+a_{n-1}x^{n-1}+...+a_{1}x+a_{0}\in\mathbb{R}[x]$ is a monic polynomial of degree n such that $p(x)>-\frac{1}{2^{n-1}}$ forall x, $-1\le x\le 1$. Prove that there exists x_{0}, $-1\le x_{0}\le 1$ such that $p(x_{0})\ge\frac{1}{2^{n-1}}$.