Taiwan National Olympiad 1995

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Day 1

1 Let $P(x)=a_0+a_1x+...+a_nx^n\in\mathbb{C}[x]$, where $a_n=1$. The roots of $P(x)$ are $b_1,b_2,...,b_n$, where $|b_1|,|b_2|,...,|b_j|>1$ and $|b_{j+1}|,...,|b_n|\le 1$. Prove that $\prod_{i=1}^{j}|b_i|\le\sqrt{|a_0|^2+|a_1|^2+...+|a_n|^2}$.

2 Given a sequence of eight integers $x_1,x_2,...,x_8$ in a single operation one replaces these numbers with $|x_1-x_2|,|x_2-x_3|,...,|x_8-x_1|$. Find all the eight-term sequences of integers which reduce to a sequence with all the terms equal after finitely many single operations.

3 Suppose that n persons meet in a meeting, and that each of the persons is acquainted to exactly 8 others. Any two acquainted persons have exactly 4 common acquaintances, and any two non-acquainted persons have exactly 2 common acquaintances. Find all possible values of n.

Day 2

4 Let $m_1,m_2,...,m_n$ be mutually distinct integers. Prove that there exists a $f(x)\in\mathbb{Z}[x]$ of degree n satisfying the following two conditions:
a)$f(m_i)=-1\forall i=1,2,...,n.$
b)$f(x)$ is irreducible.

5 Let P be a point on the circumcircle of a triangle $A_1A_2A_3$, and let H be the orthocenter of the triangle. The feet B_1,B_2,B_3 of the perpendiculars from P to A_2A_3,A_3A_1,A_1A_2 lie on a line. Prove that this line bisects the segment PH.

6 Let a,b,c,d are integers such that $(a,b)=(c,d)=1$ and $ad-bc=k>0$. Prove that there are exactly k pairs (x_1,x_2) of rational numbers with $0\le x_1,x_2<1$ for which both ax_1+bx_2,cx_1+dx_2 are integers.