Taiwan National Olympiad 1996

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Day 1

1 Suppose that a,b,c are real numbers in $(0,\frac{\pi}{2})$ such that $a+b+c=\frac{\pi}{4}$ and $\tan a=\frac{1}{x},\tan b=\frac{1}{y},\tan c=\frac{1}{z}$, where x,y,z are positive integer numbers. Find x,y,z.

2 Let $0<a\leq 1$ be a real number and let $a\leq a_{i}\leq\frac{1}{a_{i}}\forall i=\overline{1,1996}$ are real numbers. Prove that for any nonnegative real numbers $k_{i}(i=1,2,...,1996)$ such that $\sum_{i=1}^{1996}k_{i}=1$ we have $(\sum_{i=1}^{1996}k_{i}a_{i})(\sum_{i=1}^{1996}\frac{k_{i}}{a_{i}})\leq (a+\frac{1}{a})^{2}$.

3 Let be given points A,B on a circle and let P be a variable point on that circle. Let point M be determined by P as the point that is either on segment PA with $AM=MP+PB$ or on segment PB with $AP+MP=PB$. Find the locus of points M.

Day 2

4 Show that for any real numbers $a_{3},a_{4},...,a_{85}$, not all the roots of the equation $a_{85}x^{85}+a_{84}x^{84}+...+a_{3}x^{3}+3x^{2}+2x+1=0$ are real.

5 Dertemine integers $a_{1},a_{2},...,a_{99}=a_{0}$ satisfying $|a_{k}-a_{k-1}|\geq 1996$ for all $k=1,2,...,99$, such that $m=\max_{1\leq k\leq 99}|a_{k}-a_{k-1}|$ is minimum possible, and find the minimum value m^{*} of m.

6 Let $q_{0},q_{1},...$ be a sequence of integers such that

a) for any $m>n$ we have $m-n\mid q_{m}-q_{n}$, and
b) $|q_{n}|\leq n^{10},\ \forall n\geq 0$.

Prove there exists a polynomial Q such that $q_{n}=Q(n),\ \forall n\geq 0$.