Taiwan National Olympiad 1997

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Day 1

1 Let a be rational and b,c,d are real numbers, and let $f:\mathbb{R}\rightarrow [-1.1]$ be a function satisfying $f(x+a+b)-f(x+b)=c[x+2a+[x]-2[x+a]-[b]]+d$ for all x. Show that f is periodic.

2 Given a line segment AB in the plane, find all possible points C such that in the triangle ABC, the altitude from A and the median from B have the same length.

3 Let $n>2$ be an integer. Suppose that $a_{1},a_{2},...,a_{n}$ are real numbers such that $k_{i}=\frac{a_{i-1}+a_{i+1}}{a_{i}}$ is a positive integer for all i(Here $a_{0}=a_{n},a_{n+1}=a_{1}$). Prove that $2n\leq a_{1}+a_{2}+...+a_{n}\leq 3n$.

Day 2

4 Let $k=2^{2^{n}}+1$ for some $n\in\mathbb{N}$. Show that k is prime iff $k|3^{\frac{k-1}{2}}+1$.

5 Let $ABCD$ is a tetrahedron. Show that
a)If $AB=CD,AC=DB,AD=BC$ then triangles ABC,ABD,ACD,BCD are acute.
b)If the triangles ABC,ABD,ACD,BCD have the same area , then $AB=CD,AC=DB,AD=BC$.

6 Show that every number of the form $2^{p}3^{q}$, where p,q are nonnegative integers, divides some number of the form $a_{2k}10^{2k}+a_{2k-2}10^{2k-2}+...+a_{2}10^{2}+a_{0}$, where $a_{2i}\in\{1,2,...,9\}$

Day 3

7 Find all positive integers k for which there exists a function $f:\mathbb{N}\rightarrow\mathbb{Z}$ satisfying $f(1997)=1998$ and $f(ab)=f(a)+f(b)+kf(\gcd(a,b))\forall a,b$.

8 Let O be the circumcenter and R be the circumradius of an acute triangle ABC. Let AO meet the circumcircle of OBC again at D, BO meet the circumcircle of OCA again at E, and CO meet the circumcircle of OAB again at F. Show that $OD.OE.OF\geq 8R^{3}$.

9 For $n\geq k\geq 3$, let $X=\{1,2,...,n\}$ and let F_{k} a the family of k-element subsets of X, any two of which have at most $k-2$ elements in common. Show that there exists a subset M_{k} of X with at least $[\log_{2}n]+1$ elements containing no subset in F_{k}.