Taiwan National Olympiad 1998

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Day 1

1 Let m,n are positive integers.
a)Prove that $(m,n)=2\sum_{k=0}^{m-1}[\frac{kn}{m}]+m+n-mn$.
b)If $m,n\geq 2$, prove that $\sum_{k=0}^{m-1}[\frac{kn}{m}]=\sum_{k=0}^{n-1}[\frac{km}{n}]$.

2 Does there exist a solution (x,y,z,u,v) in integers greater than 1998 to the equation $x^{2}+y^{2}+z^{2}+u^{2}+v^{2}=xyzuv-65$?

3 Let m,n be positive integers, and let F be a family of m-element subsets of $\{1,2,...,n\}$ satisfying $A\cap B\neq\emptyset$ for all $A,B\in F$. Determine the maximum possible number of elements in F.

Day 2

4 Let I be the incenter of triangle ABC. Lines AI, BI, CI meet the sides of $\triangle ABC$ at D, E, F respectively. Let X, Y, Z be arbitrary points on segments EF, FD, DE, respectively. Prove that $d(X,AB)+d(Y,BC)+d(Z,CA)\leq XY+YZ+ZX$, where $d(X,\ell)$ denotes the distance from a point X to a line ℓ.

5 For a positive integer n, let $\omega(n)$ denote the number of positive prime divisors of n. Find the smallest positive tinteger k such that $2^{\omega(n)}\leq k\sqrt[4]{n}\forall n\in\mathbb{N}$.

6 In a group of $n\geq 4$ persons, every three who know each other have a common signal. Assume that these signals are not repeatad and that there are $m\geq 1$ signals in total. For any set of four persons in which there are three having a common signal, the fourth person has a common signal with at most one of them. Show that there three persons who have a common signal, such that the number of persons having no signal with anyone of them does not exceed $[n+3-\frac{18m}{n}]$.