Taiwan National Olympiad 2000

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Day 1 April 7th

1 Find all pairs (x, y) of positive integers such that $y^{x^2} = x^{y+2}$.

2 Let ABC be a triangle in which $BC < AC$. Let M be the mid-point of AB, AP be the altitude from A on BC, and BQ be the altitude from B on to AC. Suppose that QP produced meets AB (extended) at T. If H is the orthocenter of ABC, prove that TH is perpendicular to CM.

3 Consider the set $S = \{1, 2, \dots, 100\}$ and the family $\mathcal{P} = \{T \subset S \mid |T| = 49\}$. Each $T \in \mathcal{P}$ is labelled by an arbitrary number from S. Prove that there exists a subset M of S with $|M| = 50$ such that for each $x \in M$, the set $M \backslash \{x\}$ is not labelled by x.

Day 2 April 29th

1 Suppose that for some $m, n \in \mathbb{N}$ we have $\varphi(5^m - 1) = 5^n - 1$, where φ denotes the Euler function. Show that $(m, n) > 1$.

2 Let n be a positive integer and $A = \{1, 2, \dots, n\}$. A subset of A is said to be connected if it consists of one element or several consecutive elements. Determine the maximum k for which there exist k distinct subsets of A such that the intersection of any two of them is connected.

3 Define a function $f : \mathbb{N} \to \mathbb{N}_0$ by $f(1) = 0$ and

$$f(n) = \max_j \{f(j) + f(n-j) + j\} \quad \forall n \geq 2$$

Determine $f(2000)$.