Taiwan National Olympiad 2002

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Day 1

1 Find all natural numbers n and nonnegative integers $x_1, x_2, ..., x_n$ such that $\sum_{i=1}^{n} x_i^2 = 1 + \frac{4}{4n+1}(\sum_{i=1}^{n} x_i)^2$.

2 A lattice point X in the plane is said to be *visible* from the origin O if the line segment OX does not contain any other lattice points. Show that for any positive integer n, there is square $ABCD$ of area n^2 such that none of the lattice points inside the square is visible from the origin.

3 Suppose $x, y, , a, b, c, d, e, f$ are real numbers satifying
i)$\max(a,0) + \max(b,0) < x + ay + bz < 1 + \min(a,0) + \min(b,0)$, and
ii)$\max(c,0) + \max(d,0) < cx + y + dz < 1 + \min(c,0) + \min(d,0)$, and
iii)$\max(e,0) + \max(f,0) < ex + fy + z < 1 + \min(e,0) + \min(f,0)$.
Prove that $0 < x, y, z < 1$.

Day 2

4 Let $0 < x_1, x_2, x_3, x_4 \leq \frac{1}{2}$ are real numbers. Prove that $\frac{x_1x_2x_3x_4}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)} \leq \frac{x_1^4+x_2^4+x_3^4+x_4^4}{(1-x_1)^4+(1-x_2)^4+(1-x_3)^4+(1-x}$

5 Suppose that the real numbers $a_1, a_2, ..., a_{2002}$ satisfying $\frac{a_1}{2} + \frac{a_2}{3} + ... + \frac{a_{2002}}{2003} = \frac{4}{3}$ $\frac{a_1}{3} + \frac{a_2}{4} + ... + \frac{a_{2002}}{2004} = \frac{4}{5}$ ... $\frac{a_1}{2003} + \frac{a_2}{2004} + ... + \frac{a_{2002}}{4004} = \frac{4}{4005}$
Evaluate the sum $\frac{a_1}{3} + \frac{a_2}{5} + ... + \frac{a_{2002}}{4005}$.

6 Let A, B, C be fixed points in the plane , and D be a variable point on the circle ABC, distinct from A, B, C . Let I_A, I_B, I_C, I_D be the Simson lines of A, B, C, D with respect to triangles BCD, ACD, ABD, ABC respectively. Find the locus of the intersection points of the four lines I_A, I_B, I_C, I_D when point D varies.