Taiwan National Olympiad 2005

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– Written Examination

Day 1

1 Let a, b, c be three positive real numbers such that $abc = 1$. Prove that:

$$1 + \frac{3}{a+b+c} \geq \frac{6}{ab+bc+ca}.$$

2 Ten test papers are to be prepared for the National Olympiad. Each paper has 4 problems, and no two papers have more than 1 problem in common. At least how many problems are needed?

3 Let the major axis of an ellipse be AB, let O be its center, and let F be one of its foci. P is a point on the ellipse, and CD a chord through O, such that CD is parallel to the tangent of the ellipse at P. PF and CD intersect at Q. Compare the lengths of PQ and OA.

Day 2

1 P, Q are two fixed points on a circle centered at O, and M is an interior point of the circle that differs from O. M, P, Q, O are concyclic. Prove that the bisector of $\angle PMQ$ is perpendicular to line OM.

2 x, y, z, a, b, c are positive integers that satisfy $xy \equiv a \pmod{z}$, $yz \equiv b \pmod{x}$, $zx \equiv c \pmod{y}$. Prove that $\min\{x, y, z\} \leq ab + bc + ca$.

3 $a_1, a_2, ..., a_{95}$ are positive reals. Show that $\sum_{k=1}^{95} a_k \leq 94 + \prod_{k=1}^{95} \max\{1, a_k\}$

– Oral Examination

1 Let A be the sum of the first $2k+1$ positive odd integers, and let B be the sum of the first $2k+1$ positive even integers. Show that $A + B$ is a multiple of $4k+3$.

2 In triangle ABC, D is the midpoint of side AB. E and F are points arbitrarily chosen on segments AC and BC, respectively. Show that $[DEF] < [ADE] + [BDF]$.

– Additional Examination

Day 1

1 Find all integer solutions (x, y) to the equation $\frac{x+y}{x^2-xy+y^2} = \frac{3}{7}$.

2 Given a line segment $AB = 7$, C is constructed on AB so that $AC = 5$. Two equilateral triangles are constructed on the same side of AB with AC and BC as a side. Find the length of the segment connecting their two circumcenters.

3 $f(x) = x^3 - 6x^2 + 17x$. If $f(a) = 16, f(b) = 20$, find $a + b$.

Day 2

1 There are 94 safes and 94 keys. Each key can open only one safe, and each safe can be opened by only one key. We place randomly one key into each safe. 92 safes are then randomly chosen, and then locked. What is the probability that we can open all the safes with the two keys in the two remaining safes?
(Once a safe is opened, the key inside the safe can be used to open another safe.)

2 Find all reals x satisfying $0 \leq x \leq 5$ and $\lfloor x^2 - 2x \rfloor = \lfloor x \rfloor^2 - 2\lfloor x \rfloor$.

3 If positive integers p, q, r are such that the quadratic equation $px^2 - qx + r = 0$ has two distinct real roots in the open interval $(0, 1)$, find the minimum value of p.